# Homework #8 - Phys 273

1) Consider a lopsided triangle function defined from x = 0 to x = 1 meter:

$$f(x) = \begin{cases} x & 0 \le x \le d \\ \dfrac{d}{1-d}(1-x) & d \le x \le 1 \end{cases}$$

In this definition, (d) is some unitless fraction between zero and one.

a) Sketch this function (or draw it with a computer) for the case where d = 0.75.
b) This function can be represented by a Fourier Sine Series:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

where, again, L = 1 meter. According to Fourier's trick, the coefficients can be calculated:

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$

Calculate these coefficients. Hint: in the evaluation of the integral, many terms will cancel, and the correct answer comes out to be:

$$a_n = \frac{2}{(1-d)(n\pi)^2}\sin(n\pi d)$$

c) Let's stick with d = 0.75 for the remainder of this problem. Use a computer to draw the Fourier Series, but only keeping the first term in the sum.
d) Now draw the series keeping the first two terms, and the first three terms. (This amounts to two additional plots.) (Optional: keep the first 100 terms in the sum.)
e) Let's consider this lopsided triangle to be the initial state at t = 0 of a continuous string. Let the tension in the string be 10 N, and the mass density be 0.1 kg/meter. Draw the shape of the string at t = 0.005 seconds, 0.010 seconds, and 0.015 seconds, keeping the first three terms in the sum. (Optional: keep the first 100 terms in the sum.)

2) If a function f(x) is periodic, with period 2L, and if it is square integrable between (-L,L), then we can represent it as a linear combination of sine and cosine functions (a Fourier Series):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Suppose we want to represent this square wave function as a Fourier Series:

$$f(x) = \begin{cases} -1, -L < x < 0 \\ 1, 0 < x < L \end{cases}$$, periodic with period 2L.

a) Sketch this function.
b) Calculate ($a_0$) for this square wave.
c) Calculate the $\{a_n\}$, for this square wave.
d) The answer to part (c) is very simple. Why?

e) Calculate the $\{b_n\}$ for this square wave.
f) Use a plotting program to graph the Fourier Series on the interval (-3L, 3L) keeping the first three terms in the sum. (Optional: keep the first 100 terms in the sum.)

3) Let's re-calculate the Fourier Series for a square wave again, just like in problem #2, but this time let's use the complex form of the series:

$$f(x) = \sum_{n=-\infty}^{n=\infty} c_n e^{in\pi x/L}$$

Fourier's trick tells us that the coefficients $\{c_n\}$ can be calculated according to:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$$

a) Using this rule, calculate first the coefficient ($c_0$), for the square wave defined in problem #2.
b) Now calculate the rest of the Fourier coefficients $\{c_n\}$ for this square wave using the above rule..
c) Since this f(x) is purely real, the coefficients $\{c_n\}$ should have the property that

$$c_n = c_{-n}^*$$

Check to see if this is true using your result from part (b).

d) The real coefficients $\{a_n\}$ and $\{b_n\}$ that you calculated in problem #2 should be related to the complex coefficients from part (a) according to:

$$a_n = c_n + c_{(-n)}$$
$$b_n = i(c_n - c_{(-n)})$$
$$a_0 = 2c_0$$

Check to see if this is true.