Physics 273 - Homework #7

1) On Homework #6, question 4, I asked you to find the complete solution for a string loaded with three masses with a particular set of initial conditions:

1) At t = 0 the center particle (only) is displaced a distance (a) from its equilibrium position.

2) The other two particles are located at their equilibrium positions.

3) We release all three particles with an initial velocity of zero.

Now I would like you to solve the problem again, this time using Fourier's Trick:

$$a_i = \frac{\vec{y}_0 \cdot \vec{q}_i}{|\vec{q}_i|^2}$$

(Hint: since all three particles are starting from rest, we know that b_1, b_2, and b_3, the imaginary parts of the expansion coefficients, will be zero.)

2) Let's consider the continuous string with the triangular initial shape again. Remember, the string is mounted between two fixed wall at x = 0 and x = L, and its equation of motion is the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$

and the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}, \text{ where } \omega_n = \sqrt{\frac{T}{\rho}}\frac{n\pi}{L},$$

and the $\{C_n\}$ are some set of coefficients which are determined by the initial conditions. In class we calculated the real and imaginary parts of the $\{C_n\}$ for the case where the string has a triangular shape with height = (h) at t = 0:

$$y(x,t=0) = \begin{cases} \dfrac{2hx}{L} & 0 \le x \le L/2 \\ \dfrac{2h(L-x)}{L} & L/2 \le x \le L \end{cases}$$

and zero initial velocity:

$$\dot{y}(x,t=0) = 0.$$

The result was

$$\mathrm{Re}(C_n) \equiv a_n = \frac{8h}{n^2\pi^2}(-1)^{(n-1)/2} \text{ for odd (n) and } a_n = 0 \text{ for even (n), and}$$

$$\mathrm{Im}(C_n) \equiv b_n = 0 \text{ for all (n).}$$

Please turn in a plot for each of the following questions:

a) First consider the solution at t = 0. Let the initial height of the triangle be h = 0.5 meters, and the length of the string be 10 meters. Use a computer to draw the solution, but only including the first non-zero term of the infinite sum (just the first normal mode).

b) Continuing to look at the t = 0 solution, draw the solution including just the first two non-zero terms in the infinite sum.

c) Continue as in parts (a) and (b), but now including the first three non-zero terms (Optional: if it's not too much trouble, keep the first 100 non-zero terms).

d) Now keep the first three (or 100) non-zero terms, as in part (c), but this time we will allow the solution to evolve in time. Let the tension in the string be 10 N and the mass density be 0.1 kg/meter. Draw the shape of the rope at t = 0.1 seconds, keeping just the first three (or 100) terms in the sum.

e) Continue as in part (e), but now draw the shape at t = 0.2 seconds.

f) Continue as in parts (e) and (f), but now draw the shape at t = 0.3 seconds.

3) **Ortho-normality of Sine functions.** The Kronecker Delta (δ_{nm}) is defined to be equal to 0 for $n \neq m$, and equal to 1 for n = m. Given this definition, show that

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \delta_{nm}\,.$$

where (n) and (m) are integers. Explicitly evaluating the integral for

a) the n = m case.

b) the $n \neq m$ case.

Hint: You may use this trigonometric identity: $\sin(u)\sin(v) = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$.

4) The classical wave equation and its general solutions are given in problem #2. Show that the general solution is correct by explicitly substituting it into the equation of motion.

5) Consider the set of functions $\{e^{in\pi x/L}\}$, where (n) is any positive or negative integer. Show that these functions are orthogonal to each other over the interval (-L, L) by evaluating this integral:

$$\int_{-L}^{L}\left(e^{in\pi x/L}\right)\left(e^{-im\pi x/L}\right)dx\,.$$

Evaluate the integral for the case where n = m and the case where $n \neq m$.