PHYS260 Secs 0201-0205: Lecture 01: 09/03/08

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Solving the equation of motion for the simple harmonic oscillator.

This may be a bit beyond the mathematics you have had but it is straightforward to follow and is a general way to solve differential equations. We will apply it to solve the equation of motion for a harmonic oscillator,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{1}$$

To that end, we first let

$$\mathbf{D} = \frac{d}{dt}, \tag{2}$$

where we must remember that $\mathbf{D}$ is an operator, which means that $\mathbf{D}$ and x do not commute,

$$\frac{d}{dt}x \neq x\frac{d}{dt} \rightarrow \mathbf{D}x \neq x\mathbf{D}. \tag{3}$$

Now we can rewrite Eq. 1 as

$$\mathbf{D}^2x + \frac{k}{m}x = 0 \tag{4}$$

$$(\mathbf{D}^2 + \frac{k}{m})x = 0. \tag{5}$$

Factoring Eq. 5 leads to

$$\left(\mathbf{D} - i\sqrt{\frac{k}{m}}\right)\left(\mathbf{D} + i\sqrt{\frac{k}{m}}\right)x = 0, \tag{6}$$

where $i = \sqrt{-1}$. Since this is an operator Eq. 5 means that either $(\mathbf{D} - i\sqrt{\frac{k}{m}})x = 0$ or $(\mathbf{D} + i\sqrt{\frac{k}{m}})x = 0$. Taking the first, and using Eq. 2 to replace $\mathbf{D}$, we have

$$\frac{dx}{dt} = i\sqrt{\frac{k}{m}}x. \tag{7}$$

Multiplying Eq. 7 dt/x gives

$$\frac{dx}{x} = i\sqrt{\frac{k}{m}}dt. \tag{8}$$

Now, integrate the left hand side over x and the right hand side over t and we have

$$\int_x \frac{dx}{x} = i\sqrt{\frac{k}{m}}\int_t dt \tag{9}$$

$$\ln x = i\sqrt{\frac{k}{m}}t + \text{ Constant} \tag{10}$$

$$x(t) = e^{i\sqrt{\frac{k}{m}}t + \text{ Constant}} \tag{11}$$

$$= Ae^{i\sqrt{\frac{k}{m}}t}. \tag{12}$$

It is important to understand that there is a constant for both integrals that have been combined into the one on the right. The constant for the left hand side is related to the initial amplitude while that for the right hand side is related to the initial time. The last step, then, is possible by letting $A = \exp(\text{Constant})$, which we will see below is a particular equation. The function on the right in Eq. 12 is complex, which we can see if we remember the identity

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{13}$$

Since we can measure x it must be real. Thus, the real part leads to the solution

$$x(t) = \cos\left(\sqrt{\frac{k}{m}}t\right). \tag{14}$$

If we compare this to the solution given in the text we identify ω as

$$\omega = \sqrt{\frac{k}{m}} \tag{15}$$

If we are to be more correct we, actually

$$e^{\text{Constant}} = Ae^{i\phi}, \tag{16}$$

where ϕ is a phase that depends on the initial time. In the above case we have let $\phi = 0$. The more general solution will be

$$x(t) = \mathbf{Re}\left[Ae^{\sqrt{\frac{k}{m}}t+\phi}\right] \tag{17}$$

$$= A\cos\left(\sqrt{\frac{k}{m}}t + \phi\right) \tag{18}$$

$$= A\cos(\omega t + \phi). \tag{19}$$

Finally, when $\phi = 0$ the solution is given by Eq. 14 and when $\phi = \pi/2$ the cos is replaced by sin.