Mathematical description of SHM

$$F = -kx = \underbrace{ma}_{\text{Newton}}$$

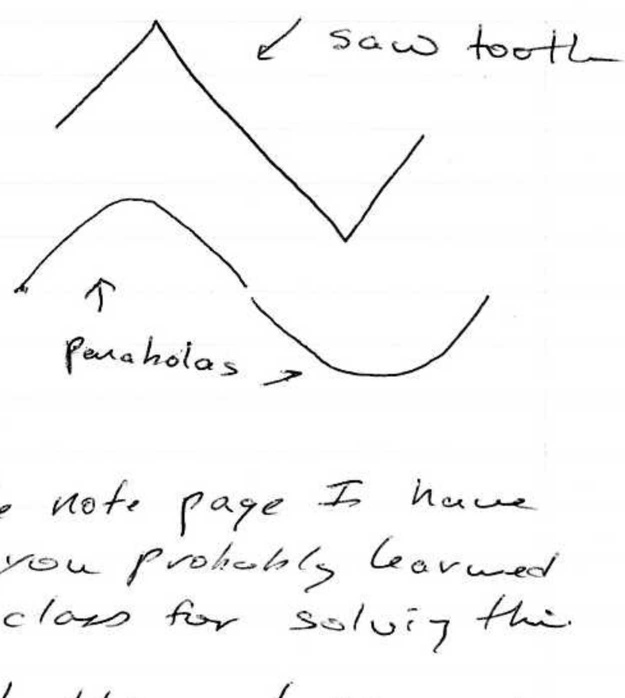
$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

We must solve this differential Eq.

Now last time we asserted that

The motion was sinusodial now we need to check this.

Could be



If you look at the note page I have posted a technique you probably learned in you calculus class for solving this

Here we will do this a different way by guessing a solution.

What do we know

Say $x = \alpha t^3$

$$\frac{d^2x}{dt^2} = 6\alpha t$$

but then
$$\underbrace{-\frac{k}{m}\overset{x}{(\cancel{\alpha} t^3)} = 6\cancel{\alpha} t}_{\text{does not work!}}$$

$$x = e^{\alpha t} \qquad \frac{d^2x}{dt} = \alpha^2 e^{+\alpha t}$$

$$-\frac{k}{m}(\cancel{e^{\alpha t}}) = \alpha^2 \cancel{e^{\alpha t}}$$

$$\alpha = \sqrt{-\frac{k}{m}} = i\sqrt{\frac{k}{m}}$$

but
$$e^{i\sqrt{\frac{k}{m}}t} = \cos\sqrt{\frac{k}{m}}\,t + i\sin\sqrt{\frac{k}{m}}\,t$$

This holds some promise

Let's let $x = A\cos\beta t$ $\qquad \beta = \sqrt{\frac{k}{m}}$

$\qquad\qquad A\sin\beta t$

$$\frac{dx}{dt} = -A\beta\sin\beta t$$
$$\qquad A\beta\cos\beta t$$

$$\frac{d^2x}{dt^2} = \left.\begin{matrix} -A\beta^2\cos\beta t \\ -A\beta^2\sin\beta t \end{matrix}\right\} \text{ has all the right properties}$$

QED

Why is there more than one solution.

Newton's equation must account for all possible scenarios that could occur.

For Ex.

I. at $t=0$ we pull the mass to the right

$\Rightarrow x(t=0) = x_0$ so $A = x_0$

$v(t=0) = 0$

$a(t=0) = -\frac{k}{m} x_0$

In this case $x = x_0 \cos\sqrt{\frac{k}{m}}\, t$

II at $t=0$ mass is at equilibrium but we give it an ~~implulse so~~ impulse so it has a velocity to the right (+ direct)

$\Rightarrow x(t=0) = 0$

$v(t=0) = v_0$

$a(t=0) = 0$ because $x=0$

$x = A \sin\sqrt{\frac{k}{m}}\, t$

we would have to differentiate x once to find A or x_0

Determining which expression to use allways require us to consider the

Boundary Conditions

General solution

$$x(t) = A\cos\sqrt{\tfrac{k}{m}}\,t + B\sin\sqrt{\tfrac{k}{m}}\,t$$

we also know that

$$\cos(\theta + \tfrac{\pi}{2}) = -\sin\theta$$

so another form of the general solution would be

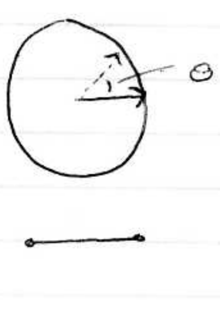
$$x(t) = \begin{matrix} A\cos(\sqrt{\tfrac{k}{m}}\,t + \phi) \\ B\sin(\sqrt{\tfrac{k}{m}}\,t + \phi) \end{matrix}$$

when ϕ is an arbitrary phase that depends on the boundary conditions

$\sqrt{\frac{k}{m}}$ appears all the time so instead of writing it in the cumbersome way we give it a special symbol

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

Now we already know $\sqrt{\frac{m}{k}}$ has dimensions of time "sec" so why not use $\nu = \frac{1}{T}$



on the unit circle ~~the head of the arrow must~~ one complete revolution is 2π rad

We already saw last time that oscillatory motion is related to uniform circular motion.

While the time it takes to sweep out the circle is T or $\frac{1}{\nu}$, the angular distance is 2π

So

ω is the angular frequency

$$\omega = 2\pi \underbrace{\nu}_{\text{linear frequency}} = \frac{2\pi}{T}$$

$$\boxed{\omega = \sqrt{\frac{k}{m}} \text{ is the angular frequency.}}$$

$$\therefore \quad \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

Note when you want to know the period can not forget this factor of 2π!!

We ~~often refer~~ often use Hz to refer to linear ~~freq~~ frequency

For angular frequency we like to use

$$\boxed{\text{rad/s}}$$

But! This is not universal!

Effective spring constants

Say you have a complicated set of forces that ~~leads~~ ~~to~~ lead to

$$F_{restore} = - k_{eff}\, x$$

all the same rules apply

$$\omega = \sqrt{\frac{k_{eff}}{m}}$$

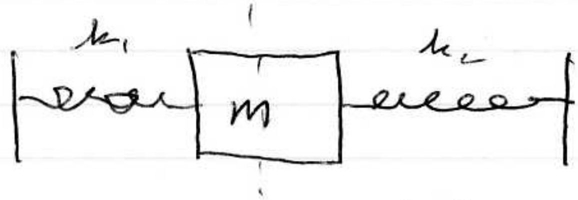
Ex. trapping atom w/ light and magnetic fields.

Magneto-optical trap ~~is~~ is a "harmonic" trap. because The leading force term is $-k_{eff}\, x$

Where k_{eff} is determined by ~~the~~ laser light interacting w/ atoms and magnetic field interaction w/ atoms as well.

$$\boxed{\text{In General} \qquad \omega = \sqrt{\frac{\text{restoring force term}}{\text{inertia term}}}}$$

Example

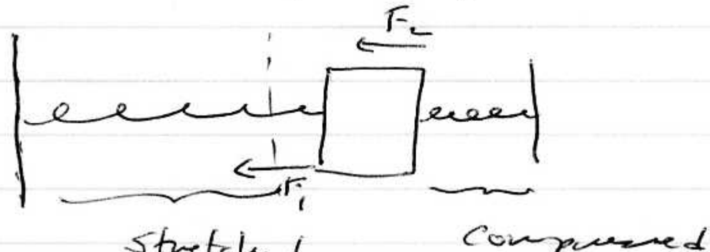


$x=0$ both spring at equilibrium

$m = 1.5 kg$ $k_1 = 20.0 N/m$ $k_2 = 15.0 N/m$

at $t=0$ we displace cart to right and release it from rest.

Approach Free Body Diagram



$$F = \sum F_i = F_1 + F_2 = -k_1 x_1 - k_2 x_2 = -(k_1 + k_2) x$$

$$k_{eff} = k_1 + k_2 = 35 N/m$$

$$\omega = \overset{4.83}{\cancel{2.33}} \text{ rad/s}$$

$$T = \frac{2\pi}{\omega} = 1.30 \text{ s}$$