Physics for Scientists and Engineers

Chapter 13

Newton's Theory of Gravity

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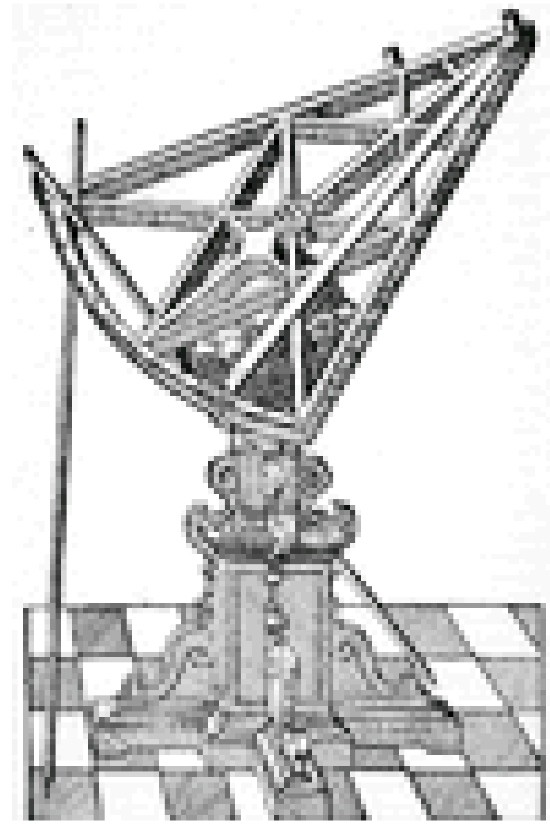
Observations of Planetary Motion



- Brahe used instruments that he invented to make accurate observations of planetary motion
  - Tycho Brahe (1546-1601), a Danish nobleman, was the last of the "naked eye" astronomers
- Kepler analyzed Brahe's data and formulated three laws of planetary motion
  - Johannes Kepler (1571-1630), a German astronomer, was Brahe's assistant

Kepler's Laws

- Kepler's 1st Law:

  All planets move in *elliptical orbits* with the Sun at one focus

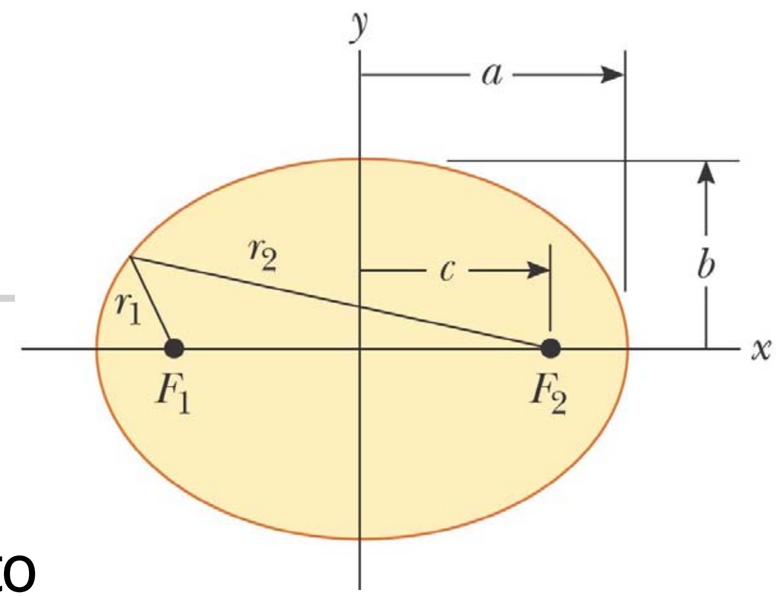
- Kepler's 2nd Law:

  The radius vector drawn from the Sun to a planet sweeps out *equal areas* in equal time intervals

- Kepler's 3rd Law:

  The *square* of the orbital period of any planet is proportional to the *cube* of the semimajor axis of the elliptical orbit

Notes on Ellipses



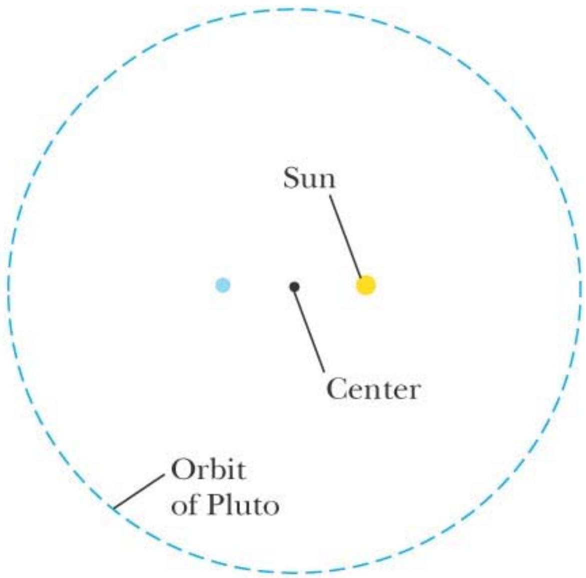
- F_1 and F_2 are each a *focus* of the ellipse
- The sum of lengths from the foci to any point on the ellipse is a constant, i.e. $r_1 + r_2$ = constant
- The longest distance through the center is the *major axis*
  - *a* is the *semimajor axis*
- The shortest distance through the center is the *minor axis*
  - *b* is the *semiminor axis*
- The *eccentricity* of the ellipse is defined as $e = c/a$
  - For a circle, $e = 0$
  - The range of values of the eccentricity for ellipses is $0 < e < 1$

Kepler's First Law

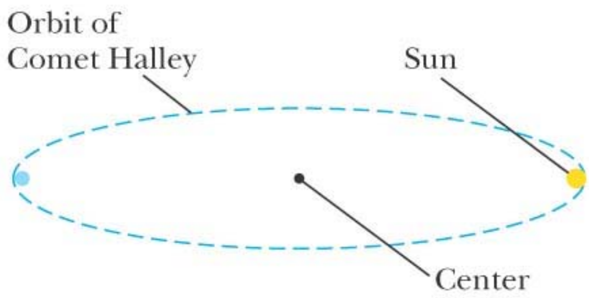
- Kepler's 1st Law is a direct result of the *inverse square* nature of the gravitational force
- *Elliptical* orbits are allowed for *bound* objects
  - A bound object repeatedly orbits the center
  - A *circular* orbit is a special case of the general elliptical orbits
- *Unbound* objects could have paths that are *parabolas* ($e = 1$) and *hyperbolas* ($e > 1$)
  - An unbound object would pass by and not return

Orbit Examples

- Earth's orbit has a very small eccentricity
  - $e_{\text{Earth}} = 0.0167$
- Pluto has the highest eccentricity of any planet
  - $e_{\text{Pluto}} = 0.25$
- Halley's comet has an orbit with a very high eccentricity
  - $e_{\text{Halley's comet}} = 0.97$



(a)

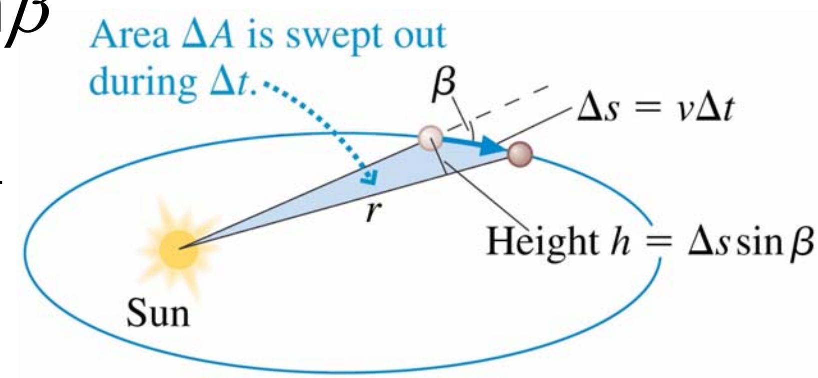


(b)

Kepler’s Second Law

- Kepler’s 2nd Law is a consequence of the *conservation of angular momentum*
  - Angular momentum is conserved because there is no tangential force
- Geometrically, in time *dt*, the radius vector sweeps out the area $\Delta A = \frac{1}{2} r v \Delta t \sin\beta$

$$\frac{dA}{dt} = \frac{1}{2} r v \sin\beta = \frac{m r v \sin\beta}{2m} = \frac{L}{2m}$$
$$= \text{constant}$$



  - The law applies to any central force, whether $1/r^2$ or not

Kepler's Third Law

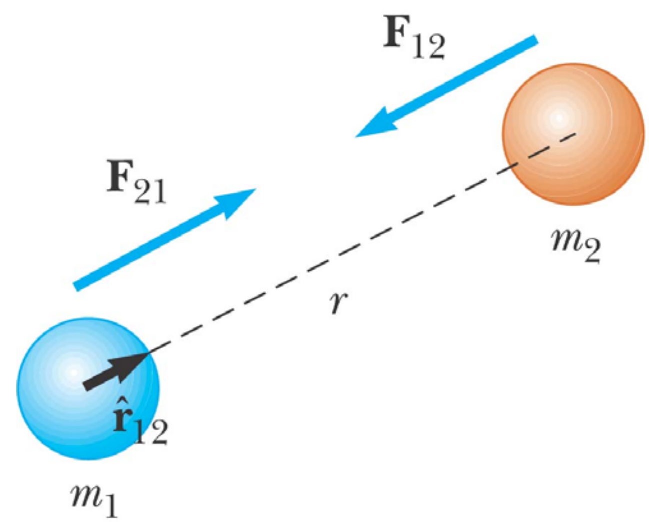
- Kepler's 3rd Law is a consequence of the *inverse square law*
- For most planets, the orbits are nearly circular and we can then use the *radius* of the orbit rather than the *semimajor axis* to prove the Third Law
  - The gravitational force supplies a centripetal force

$$F_g = \frac{GM_{Sun} M_{Planet}}{r^2} = M_{Planet} \frac{v^2}{r} \Rightarrow v^2 = \frac{GM_{Sun}}{r}. \quad \text{Also, } T = \frac{2\pi r}{v}$$

$$\Rightarrow T^2 = \left(\frac{4\pi^2}{GM_{Sun}} \right) r^3. \quad \text{More generally, } T^2 = \left(\frac{4\pi^2}{GM_{Sun}} \right) a^3$$

Newton's Law of Gravitation

- *Every* particle in the Universe *attracts every other* particle with a force proportional to the product of their masses and *inversely* proportional to the distance between them

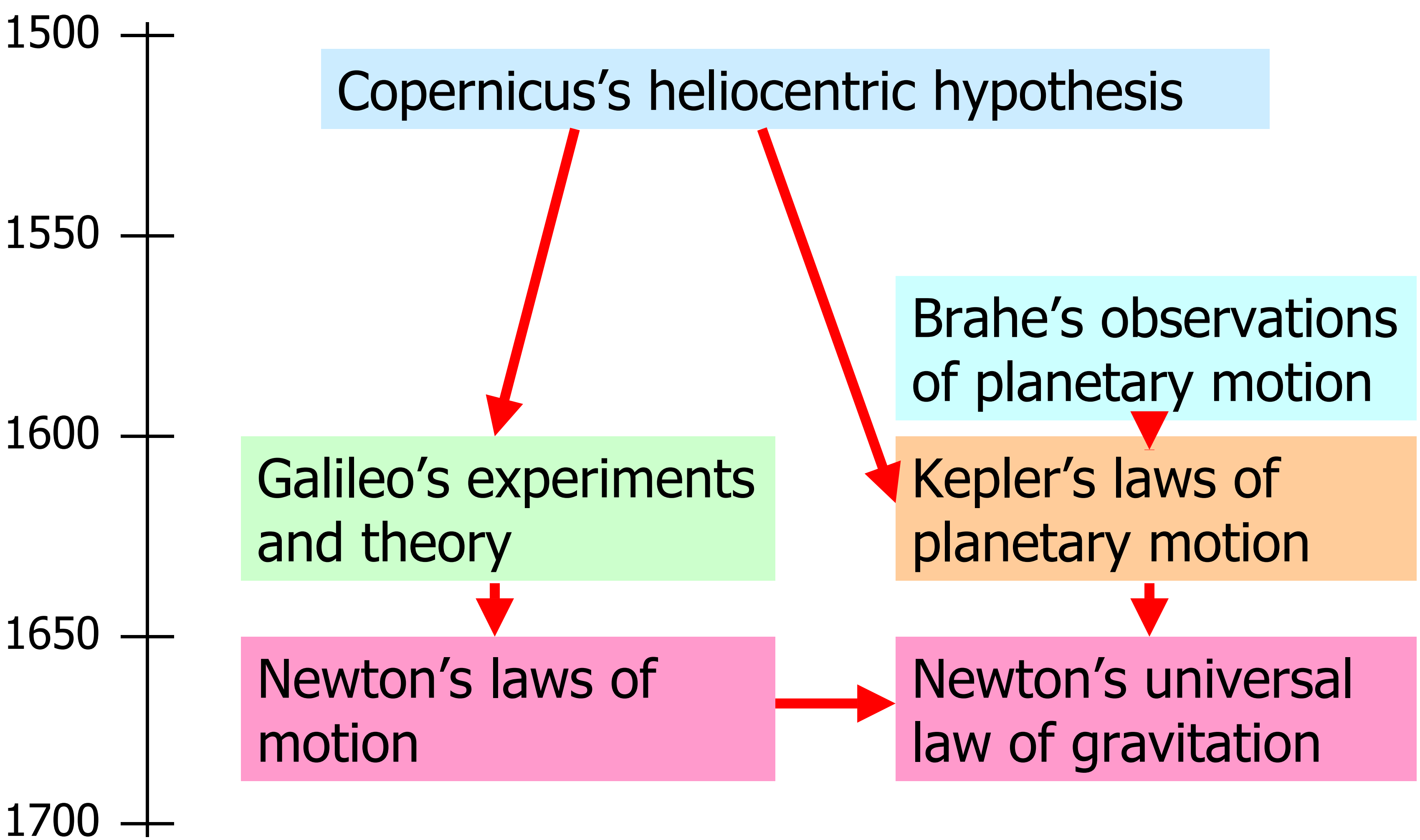


- In vector form,

$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}_{12}$$

  - $G = 6.674 \times 10^{-11}$ N·m^2/kg^2 is the *universal gravitational constant*
  - $\mathbf{F}_{12} = -\mathbf{F}_{21}$: Newton's 3rd Law action-reaction pair

Discovery of Newton’s Laws



Apple and the Moon

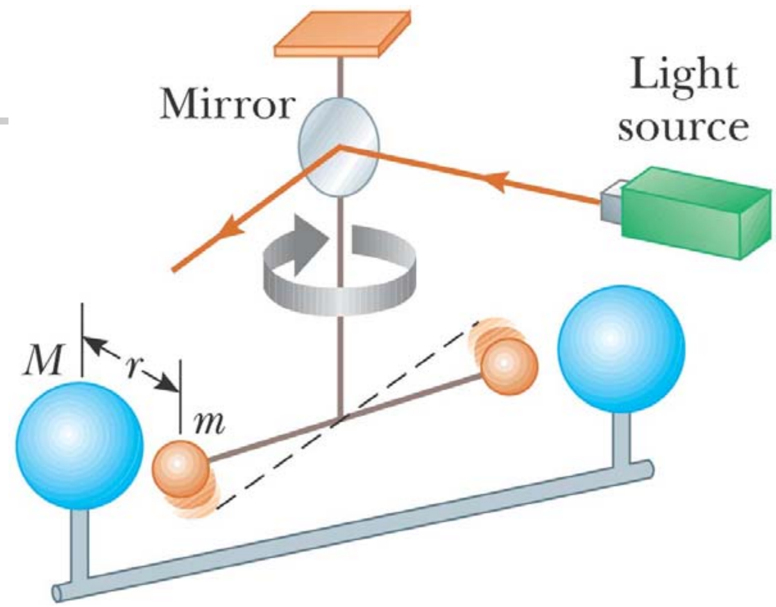
- Newton compared the acceleration of the Moon in its orbit with the acceleration of a falling apple
  - Moon: $g_M = \frac{v_M^2}{r_M} = \frac{4\pi^2 r_M}{T_M^2} = \frac{4\pi^2 (3.84 \times 10^8 \text{ m})}{(27.3 \text{ days})^2} = 0.00272 \text{ m/s}^2$
  - Apple: $g_a = 9.80 \text{ m/s}^2$
  - Ratio of forces: $\frac{g_M}{g_a} = \frac{0.00272 \text{ m/s}^2}{9.80 \text{ m/s}^2} = \frac{1}{3600} = \left(\frac{R_E}{r_M}\right)^2$
- But it took Newton 22 years to publish the $1/r^2$ law
  - He needed to develop calculus to show that a sphere behaves like a point mass at its CM

G versus g

- Always distinguish between *G* and *g*
- *G* is the *universal gravitational constant*
  - It is the same everywhere
- *g* is the magnitude of *gravitational field*, i.e. the gravitational force per unit mass
  - $g = 9.80\ m/s^2$ average at the surface of the Earth
  - *g* will vary by location
  - *g* differs for each planet

Measuring *G*

- *G* was first measured by Henry Cavendish in 1798
- The *torsion balance* shown here allowed the attractive force between two spheres to cause the rod to rotate
  - The mirror amplifies the motion
  - It was repeated for various masses



Torsion Balance

- *G* is the *least* well-known constant of nature

Finding *g* from *G*

- The magnitude of the force acting on an object of mass *m* in free fall near the Earth's surface is *mg*
- This can be set equal to the force of universal gravitation acting on the object

$$mg = G\frac{M_E m}{R_E^2} \quad \Rightarrow \quad g = G\frac{M_E}{R_E^2}$$

- You can "weigh" the Earth using values of *g* and *G*

$$M_E = \frac{gR_E^2}{G} = \frac{(9.80\ \text{m/s}^2)(6.37\times10^6\ \text{m})^2}{6.67\times10^{-11}\ \text{N m}^2/\text{kg}^2} = 5.98\times10^{24}\ \text{kg}$$

Problem 1: Planet Mass

As an astronaut, you observe a small planet to be spherical. After landing on the planet, you set off, walking always straight ahead, and find yourself returning to your spacecraft from the opposite side after completing a lap of 25.0 km. You hold a hammer and a feather at a height of 1.40 m, release them, and observe that they fall together to the surface in 29.2 s.

Determine the mass of the planet.

From the walk, $2\pi R = 25000\ \text{m}.\ R = \dfrac{25000\ \text{m}}{2\pi} = 3.98\times10^{3}\ \text{m}$

From the drop, $\Delta y = \dfrac{1}{2}gt^2 = \dfrac{1}{2}g(29.2\ \text{s})^2 = 1.40\ \text{m},\ g = \dfrac{2(1.40\ \text{m})}{(29.2\ \text{s})^2} = 3.28\times10^{-3}\ \text{m/s}^2 = \dfrac{GM}{R^2}$

$$M = \frac{gR^2}{G} = \frac{(3.28\times10^{-3}\ \text{m/s}^2)(3.98\times10^{3}\ \text{m})^2}{6.67\times10^{-11}\ \text{N m}^2/\text{kg}^2} = 7.79\times10^{14}\ \text{kg}$$

g Above the Earth's Surface

- If an object is some distance h above the Earth's surface, r becomes $R_E + h$

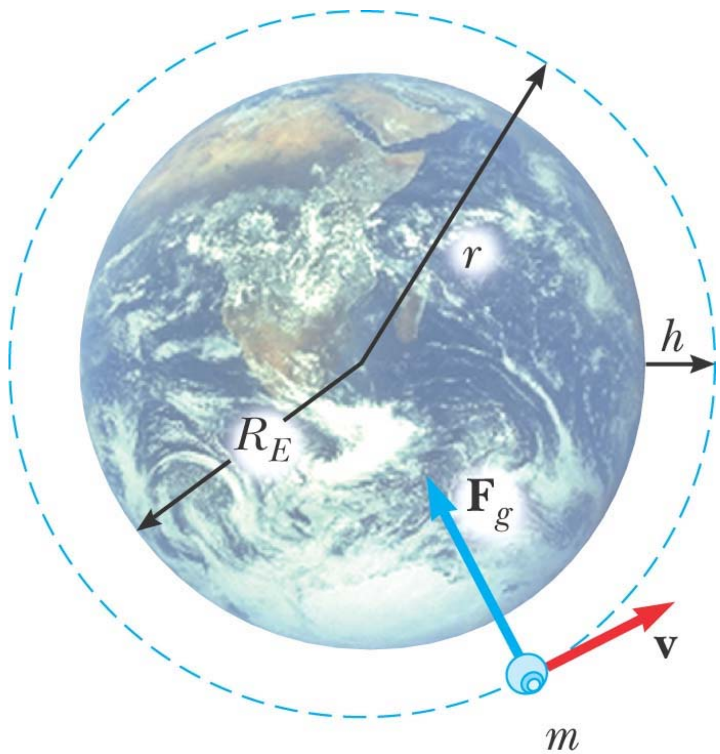
$$g = \frac{GM_E}{(R_E + h)^2}$$

  - g decreases with increasing altitude
  - As $r \to \infty$, the weight of the object approaches zero

Altitude h (km)	g (m/s^2)
1 000	7.33
2 000	5.68
3 000	4.53
4 000	3.70
5 000	3.08
6 000	2.60
7 000	2.23
8 000	1.93
9 000	1.69
10 000	1.49
50 000	0.13
∞	0

Geosynchronous Satellite

- A geosynchronous satellite remains over the same point on the Earth
- From Kepler's 3rd Law, we can find the *h* for which the satellite has a period of 1 day



$$T^2 = \left(\frac{4\pi^2}{GM_E}\right)(R_E + h)^3$$

$$(24\text{ hr})^2 = \left(\frac{4\pi^2}{6.67\times10^{-11}\times5.98\times10^{24}}\right)(6.37\times10^6\text{ m} + h)^3 \Rightarrow h = 35.8\times10^6\text{ m}$$

The Gravitational Field

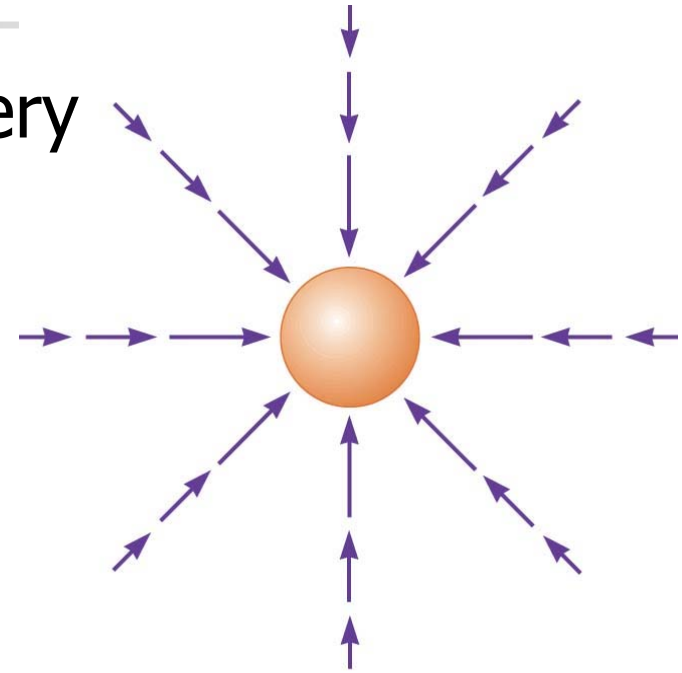
- The *gravitational field* is the gravitational *force* experienced by a test particle placed at that point *divided by the mass* of the test particle

$$\mathbf{g} = \frac{\mathbf{F}_g}{m} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

- When a particle of mass *m* is placed at a point where the gravitational field is **g**, the particle experiences a force $\mathbf{F}_g = m\mathbf{g}$
  - **g** does not necessarily have the magnitude of 9.80 m/s^2

The Gravitational Field, cont

- A gravitational field exists at every point in space
  - Points in the *direction* of the acceleration a particle would experience, if placed in that field
  - The *magnitude* is that of the free-fall acceleration at that location



- The gravitational field describes the *effect* that any object has on the empty space around itself in terms of the force that *would* be present *if* a second object were somewhere in that space

Gravitational Potential Energy

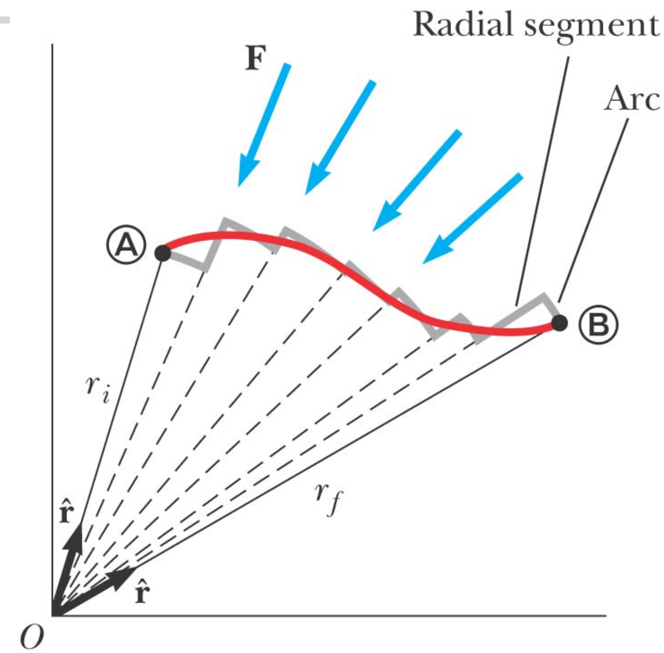
- The work done by **F** along any segment is

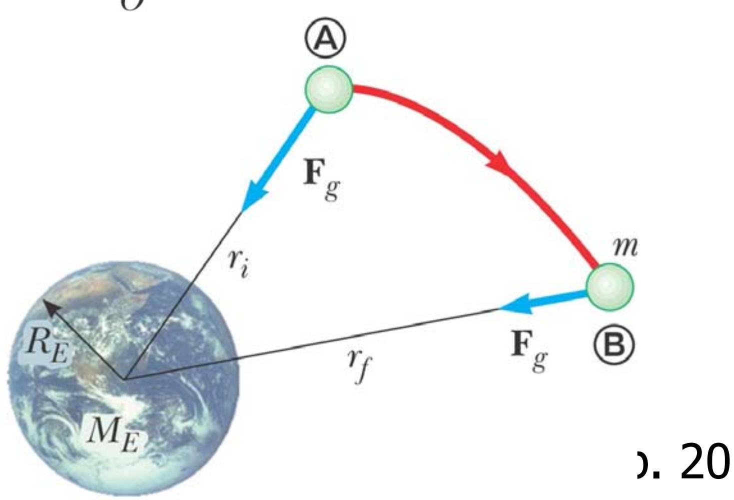
$$dW = \mathbf{F} \cdot d\mathbf{r} = F(r)dr$$

- The total work is

$$W = \int_{r_i}^{r_f} F(r)dr = -\int_{r_i}^{r_f} \frac{GM_E m}{r^2} dr$$

$$= GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$





Gravitational Potential Energy, cont

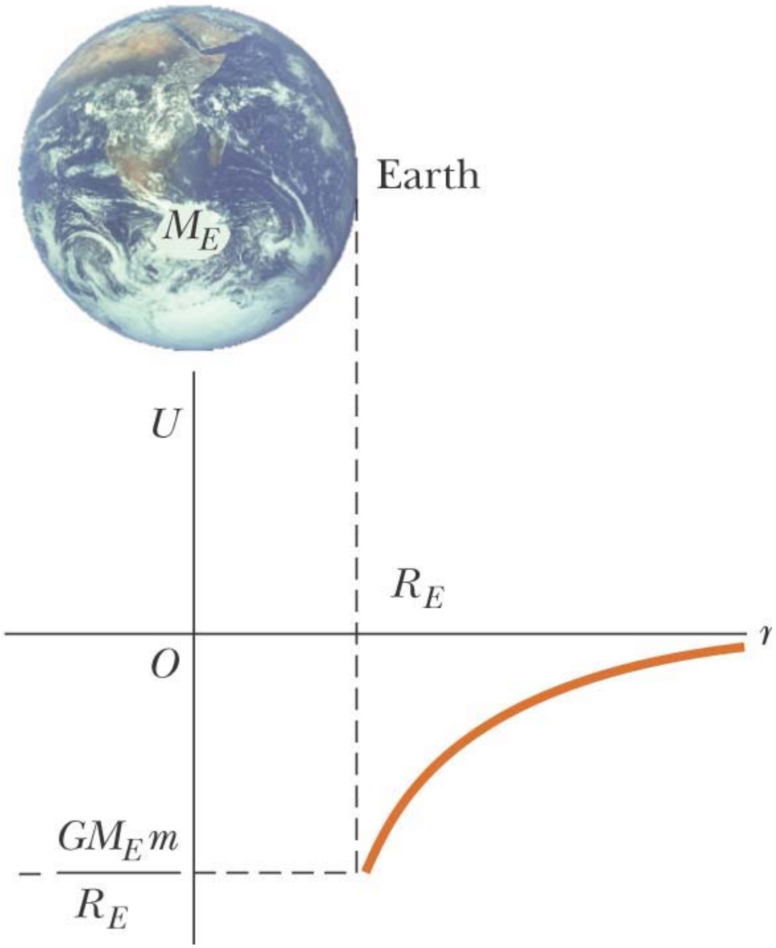
- ΔU is the negative of the work done

$$U_f - U_i = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

- We associate the gravitational potential energy with

$$U(r) = -\frac{GM_E m}{r}$$

  - $U(r)$ =0 is chosen at $r = \infty$



Gravitational Potential Energy, cont

- For any two particles, the gravitational potential energy becomes

$$U = -\frac{Gm_1m_2}{r}$$

  - The gravitational potential energy between any two particles varies as $1/r$ while the force varies as $1/r^2$
  - The potential energy is *negative* because the force is *attractive* and we chose the potential energy to be zero at infinite separation
  - An external agent must do *positive* work to *increase the separation* between two objects

Grav Potential Energy Near Earth

- At a small distance *h* above the Earth's surface,

$$U = -\frac{GM_E m}{R_E + h} = -\frac{GM_E m}{R_E(1 + h/R_E)} = \frac{-mgR_E}{(1 + h/R_E)}$$

$$\text{For small } x, \quad (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 \ldots\ldots$$

$$U = -mgR_E\left(1 + \frac{h}{R_E}\right)^{-1} = -mgR_E\left[1 - \frac{h}{R_E} + \left(\frac{h}{R_E}\right)^2 + \ldots\right]$$

$$U \approx -mgR_E + mgh$$

- The potential difference between two points at h and $h + y$ is

$$\Delta U = mg(h+y) - mgh = mgy$$

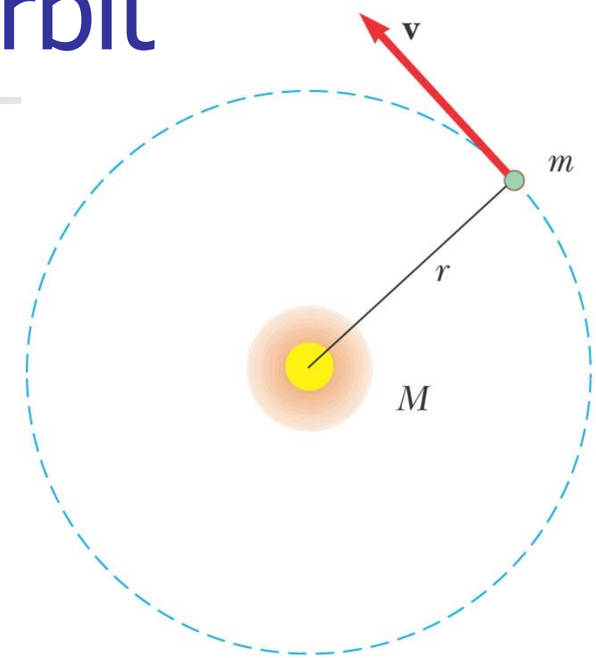
Energy and Satellite Motion

- Assume an object of mass m moving with a speed v in the vicinity of a massive object of mass M (>> m)
  - Also assume M is at rest in an inertial frame
- The total energy is the sum of the system's kinetic and potential energies

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

  - In a bound system, E is necessarily less than 0

Energy in a Circular Orbit



- The gravitational force causes a centripetal acceleration

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

- The energies are related by

$$U = -\frac{GMm}{r}, \; K = \frac{1}{2}mv^2 = \frac{GMm}{2r} = -\frac{U}{2}$$

$$E = K + U = +\frac{U}{2} = -\frac{GMm}{2r} < 0$$

- In an elliptical orbit,

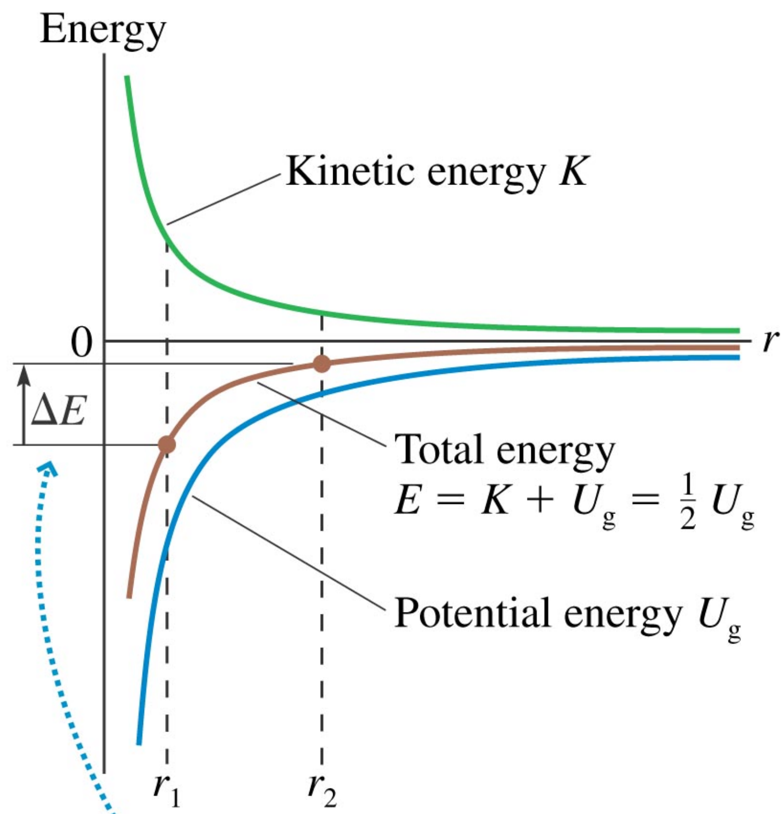
$$E = -\frac{GMm}{2a}$$

Satellite Orbit Transfer

- To raise a satellite from a lower altitude (r_1) circular orbit to a higher altitude (r_2) one, energy must be provided by the amount

$$\Delta E = \frac{GMm}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

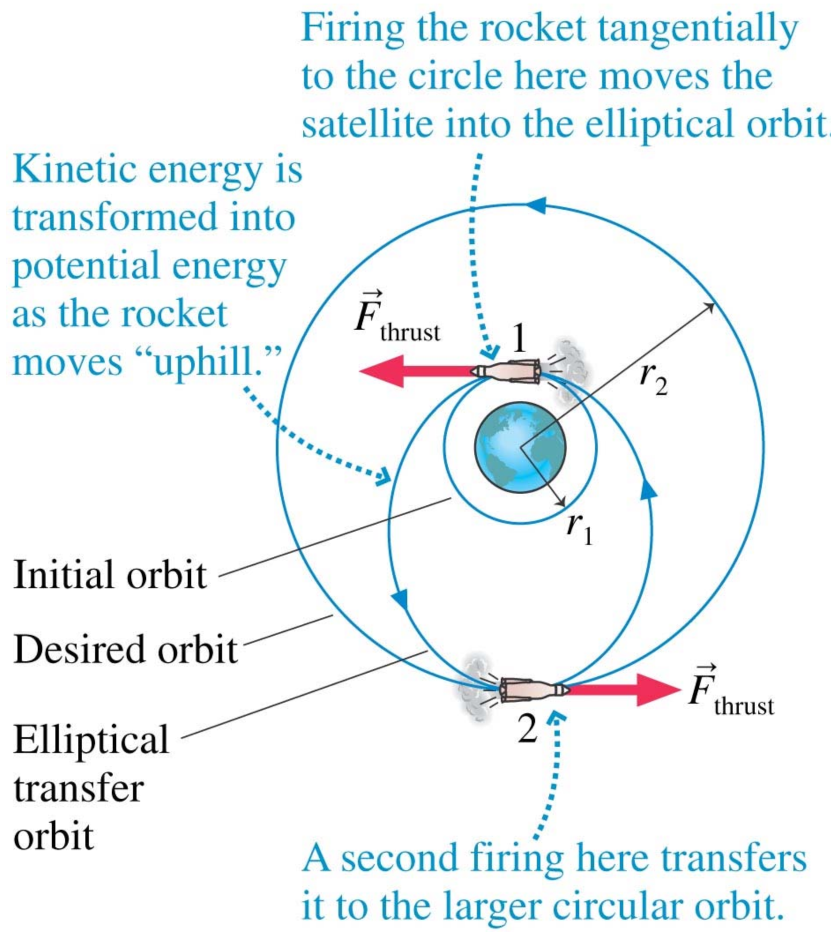
  - The satellite must climb "up hill"



Energy ΔE must be added to move a satellite from an orbit with radius r_1 to radius r_2.

Satellite Orbit Transfer, cont

- A *forward* thruster is fired to increase the kinetic energy and put the satellite into an *elliptical* orbit
- Upon reaching the desired altitude, a second firing of a *forward* thruster *circularizes* the orbit
  - The kinetic energy increases to satisfy $K = -\frac{1}{2}U$

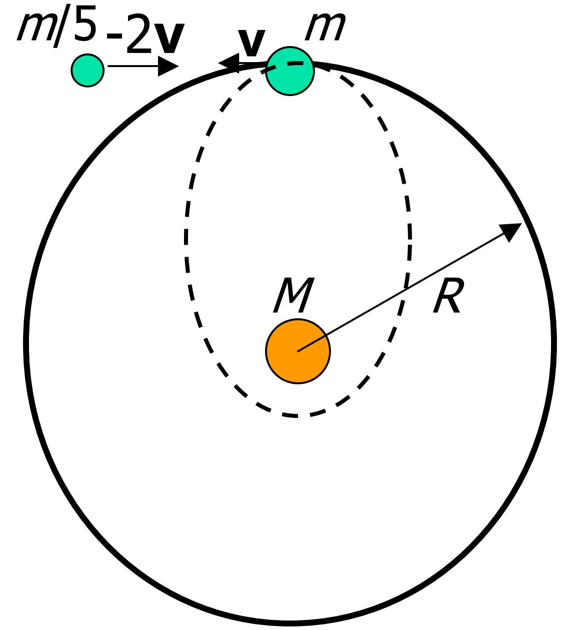


Two-Particle Bound System

- Both the *total energy* and the *angular momentum* of a two-object system are constants of the motion
  - Total energy is $E = \frac{1}{2}mv_i^2 - \frac{GMm}{r_i} = \frac{1}{2}mv_f^2 - \frac{GMm}{r_f}$
  - Angular momentum is $\mathbf{L} = \mathbf{r}_i \times m\mathbf{v}_i = \mathbf{r}_f \times m\mathbf{v}_f$
- The absolute value of *E* is the *binding energy* of the system
  - If an external agent supplies energy larger than the binding energy, the system will become unbound

Problem 2: Collision with a Comet

A comet of mass $m/5$ is making a totally inelastic head-on collision with Earth with a velocity of $-2\mathbf{v}$, where m and $\mathbf{v}$ are the mass and orbital velocity of Earth. Earth was originally in a circular orbit around the Sun with radius R. Ignore the effect of the gravitational interaction of the comet with Earth (or the Sun) before collision.



(a) What is the new orbital velocity of Earth, $\mathbf{v}'$, right after the collision? (b) Show that the new orbit of Earth around the Sun is an ellipse with $R_{max} = R$ and $R_{min} = R/7$. (c) What is the new orbital period of Earth?

Problem 2, cont

(a) Linear momentum is conserved inelastic collision.

$$m\mathbf{v} + \frac{m}{5}\left(-2\mathbf{v}\right) = \left(m + \frac{m}{5}\right)\mathbf{v}', \ \frac{3}{5}m\mathbf{v} = \frac{6}{5}m\mathbf{v}', \ \mathbf{v}' = \frac{1}{2}\mathbf{v}$$

(b) Since velocity is reduced, the orbit becomes elliptical

with $R_{\max} = R$ and $v_{\min} = v'$.

In the new orbit, angular momentum and energy are conserved :

$$mv_{\max}R_{\min} = mv_{\min}R_{\max} = \frac{1}{2}mvR, \ v_{\max} = \frac{v}{2}\frac{R}{R_{\min}}$$

$$\frac{1}{2}mv_{\max}^2 - \frac{GMm}{R_{\min}} = \frac{1}{2}mv_{\min}^2 - \frac{GMm}{R_{\max}}, \ \frac{1}{2}\left(\frac{v}{2}\frac{R}{R_{\min}}\right)^2 - \frac{GM}{R_{\min}} = \frac{1}{2}\left(\frac{v}{2}\right)^2 - \frac{GM}{R}$$

Problem 2, cont

$$\frac{1}{8}v^2\left(\frac{R^2}{R_{\min}^2}-1\right)=\frac{GM}{R}\left(\frac{R}{R_{\min}}-1\right).$$

Substituting $v^2=\dfrac{GM}{R}$, $\dfrac{R}{R_{\min}}+1=8$, $R_{\min}=\dfrac{R}{7}$

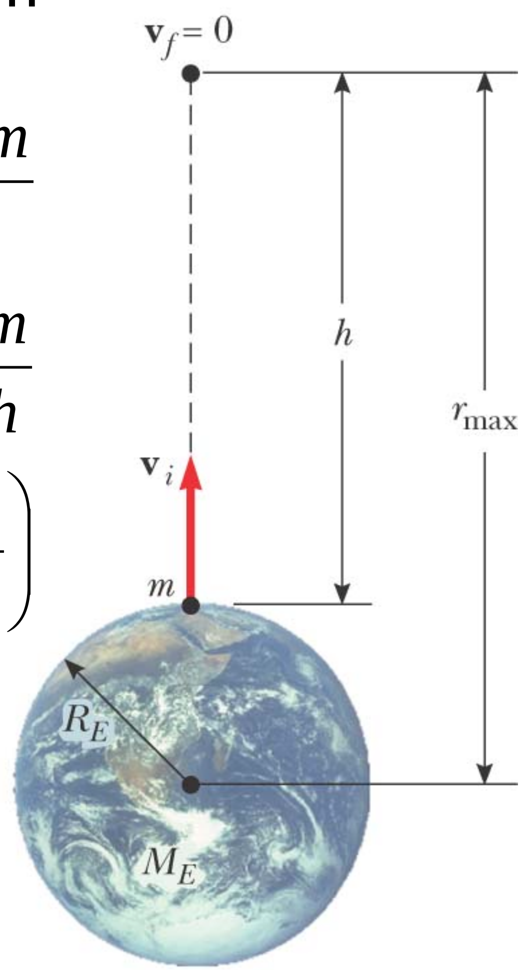
(c) The new semimajor axis is $a'=\dfrac{1}{2}\left(R_{\max}+R_{\min}\right)=\dfrac{1}{2}\left(R+\dfrac{R}{7}\right)=\dfrac{4}{7}R=\dfrac{4}{7}a.$

From Kepler's 3rd law, $\tau'=\tau\left(\dfrac{a'}{a}\right)^{3/2}=365\,\text{days}\left(\dfrac{4}{7}\right)^{2/3}=157\,\text{days}$

Escape Speed

- An object of mass m is projected upward from the Earth's surface with an initial speed v_i
  - Total energy at takeoff: $E_i = \frac{1}{2}mv_i^2 - \frac{GM_E m}{R_E}$
  - Total energy at max altitude: $E_f = -\frac{GM_E m}{R_E + h}$
  - Energy is conserved: $v_i^2 = 2GM_E\left(\frac{1}{R_E} - \frac{1}{R_E + h}\right)$
- To escape the Earth to $h = \infty$,

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$$



Escape Speed, cont

- The table gives escape speeds from various planets and the Sun
- Complete escape from an object is not really possible
  - Some gravitational force will always be felt no matter how far away you can get
- This explains why some planets have atmospheres and others do not
  - Lighter molecules have higher average speeds and are more likely to reach escape speeds

Escape Speeds from the Surfaces of the Planets, Moon, and Sun

Planet	v_{esc} (km/s)
Mercury	4.3
Venus	10.3
Earth	11.2
Mars	5.0
Jupiter	60
Saturn	36
Uranus	22
Neptune	24
Pluto	1.1
Moon	2.3
Sun	618

Problem 3: Voyager

Voyagers 1 and *2* surveyed the surface of Jupiter's moon Io and photographed active volcanoes spewing liquid sulfur to heights of 70 km above the surface of this moon.

Find the speed with which the liquid sulfur left the volcano. Io's mass is 8.90×10^{22} kg, and its radius is 1820 km.

Since mechanical energy of sulfur is conserved,

$$\frac{1}{2}mv_i^2 - \frac{GM_I m}{R_I} = 0 + \frac{GM_I m}{R_I + h}$$

$$v_i^2 = 2GM_I\left(\frac{1}{R_I} - \frac{1}{R_I + h}\right) = 2\left(6.67\times10^{-11}\right)\left(8.90\times10^{22}\right)\left(\frac{1}{1.82\times10^{6}} - \frac{1}{1.89\times10^{6}}\right)$$

$$v_i = 492 \text{ m/s}$$

Problem 4: Satellite Air Resistance

Many people assume that air resistance acting on a moving object will always make the object slow down. It can actually be responsible for making the object speed up. Consider a 100-kg Earth satellite in a circular orbit at an altitude of 200 km. A small force of air resistance makes the satellite drop into a circular orbit with an altitude of 100 km.

(a) What is the initial speed?
(b) What is the final speed?
(c) What is the initial energy?
(d) What is the final energy?
(e) What is the energy loss?
(f) What force makes the satellite's speed increase?

Problem 4, cont

(a) For both circular orbits, $\frac{GM_E m}{r^2} = \frac{mv^2}{r}$ **and** $v = \sqrt{\frac{GM_E}{r}}$

$$v_i = \sqrt{\frac{(6.67\times10^{-11}\ \mathbf{N\,m^2/kg^2})(5.98\times10^{24}\ \mathbf{kg})}{\mathbf{6.37\times10^6\ m + 2.0\times10^5\ m}}} = 7.79\times10^3\ \mathbf{m/s}$$

(b) $v_f = \sqrt{\frac{(6.67\times10^{-11}\ \mathbf{N\,m^2/kg^2})(5.98\times10^{24}\ \mathbf{kg})}{\mathbf{6.37\times10^6\ m + 1.0\times10^5\ m}}} = 7.85\times10^3\ \mathbf{m/s}$

So the satellite speeds up as it spirals down the orbit.

(c) The total energy of the satellite-Earth system is $E = K + U = -\frac{GM_E m}{2r}$

$$E_i = -\frac{(6.67\times10^{-11}\ \mathbf{N\,m^2/kg^2})(5.98\times10^{24}\ \mathbf{kg})(\mathbf{100\,kg})}{\mathbf{2(6.37\times10^6\ m + 2.0\times10^5\ m)}} = -3.04\times10^9\ \mathbf{J}$$

(d) $E_f = -\frac{(6.67\times10^{-11}\ \mathbf{N\,m^2/kg^2})(5.98\times10^{24}\ \mathbf{kg})(\mathbf{100\,kg})}{\mathbf{2(6.37\times10^6\ m + 1.0\times10^5\ m)}} = -3.08\times10^9\ \mathbf{J}$

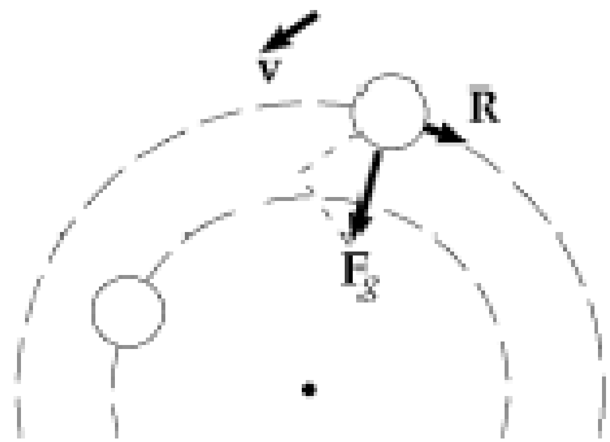
Problem 4, cont

(e) $\Delta E = E_f - E_i = (-3.08\times10^9\ \text{J}) - (-3.04\times10^9\ \text{J}) = -4.69\times10^7\ \text{J}$

The spacecraft loses energy as it spirals down the orbit.

(f) The only forces on the satellite are the force of air resistance,

$$F = \frac{1}{2} D\rho v^2 A,$$



which is comparatively small, and the force of gravity. Because the spiral path is not perpendicular to the gravitational force, the radial force pulls on the descending satellite to do positive work and make its speed increase.

Problem 5: Launching Payload

(a) Determine the amount of work that must be done on a 100-kg payload to elevate it to a height of 1000 km above the Earth's surface, i.e. without orbital motion.

(b) Determine the amount of additional work that is required to put the payload into a circular orbit at this elevation.

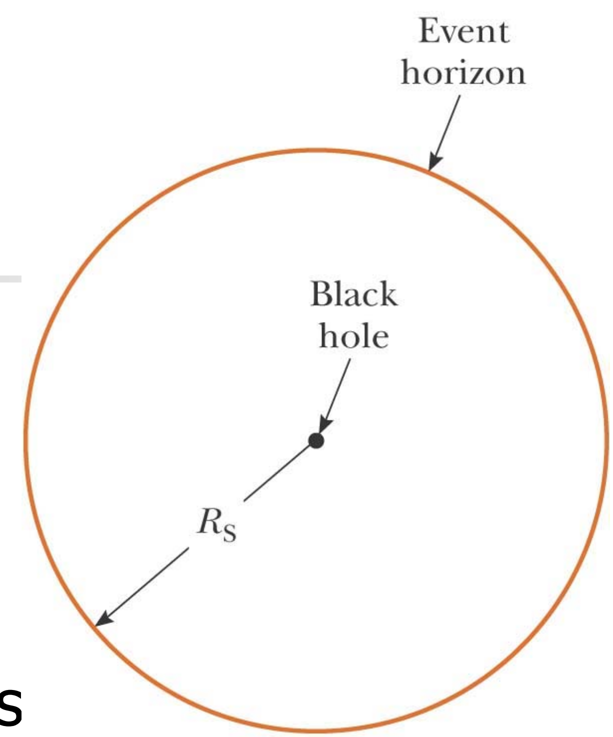
(a) $$W = U_f - U_i = -\frac{GM_E m}{r_f} + \frac{GM_E m}{r_i} = GM_E m\left(\frac{1}{R_E} - \frac{1}{R_E + y}\right)$$

$$= \left(6.67\times10^{-11}\ \frac{\text{N m}^2}{\text{kg}^2}\right)\left(5.98\times10^{24}\ \text{kg}\right)\left(100\ \text{kg}\right)\left(\frac{1}{6.37\times10^6\ \text{m}} - \frac{1}{7.37\times10^6\ \text{m}}\right) = 8.50\times10^8\ \text{J}$$

(b) An additional work must be done to provide the kinetic energy.

$$K = -\frac{U_f}{2} = \frac{GM_E m}{2(R_E + y)} = \frac{(6.67\times10^{-11}\ \text{N m}^2/\text{kg}^2)(5.98\times10^{24}\ \text{kg})(100\ \text{kg})}{2(7.37\times10^6\ \text{m})} = 2.71\times10^9\ \text{J}$$

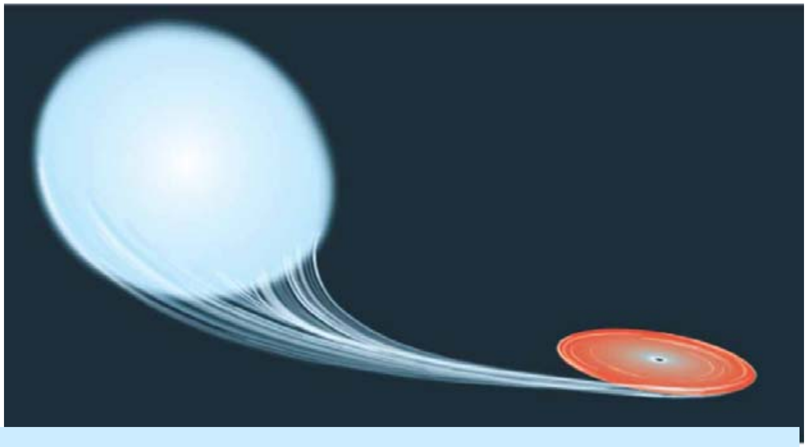
Black Holes



- A *black hole* is the remains of a star that has collapsed under its own gravitational force
- The escape speed for a BH is very large due to the concentration of a large mass into a sphere of very small radius
  - The escape speed exceeds the speed of light so radiation cannot escape and it appears black
- The critical radius at which the escape speed equals *c* is called the *Schwarzschild radius* R_S
  - The imaginary surface of a sphere with R_S is called the *event horizon*

Black Holes and Accretion Disk

- Although light from a BH cannot escape, light from events taking place near, but outside the event horizon of, the BH should be visible
  - If a binary star system has a BH and a normal star, the material from the normal star can be pulled into an *accretion disk* around the BH
  - The high-temperature material emits x-ray



An ordinary star on the left and a black hole on the right surrounded by an accretion disk

- There is evidence that supermassive BHs exist at the centers of galaxies