Physics for Scientists and Engineers

Chapter 4

Kinematics in Two Dimensions

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Ho Jung Paik

Acceleration

- *Average acceleration* of a moving object is defined as

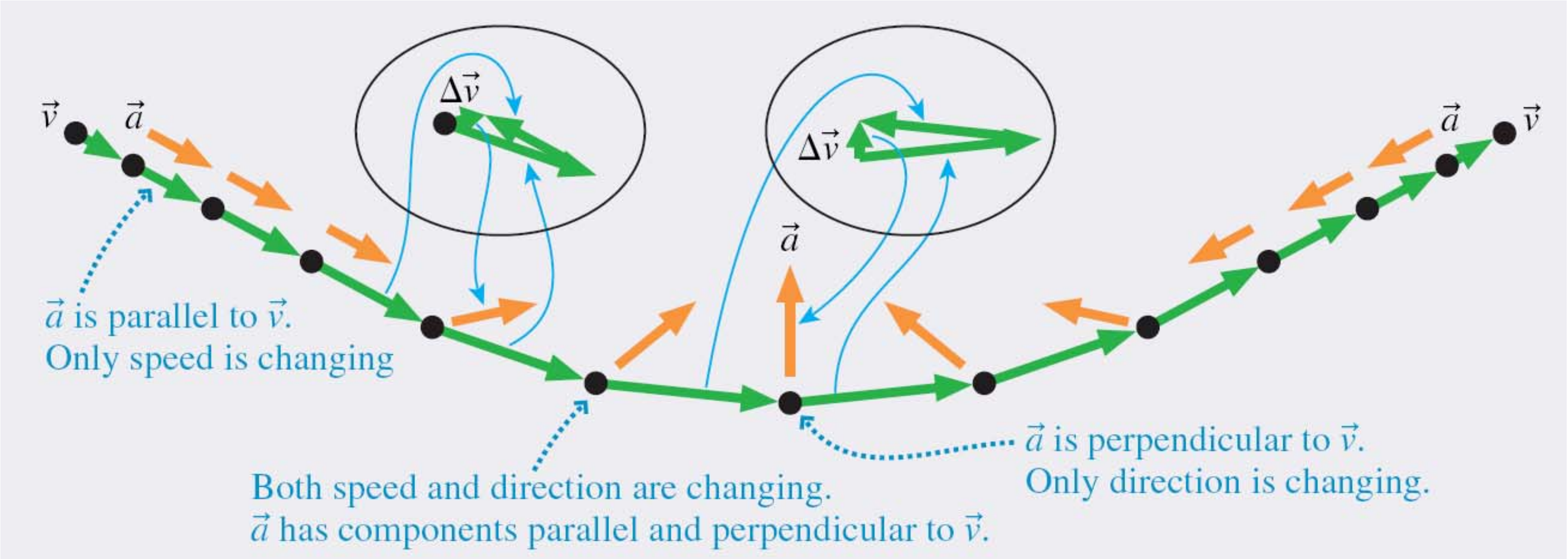
$$\vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t}$$

- As an object moves, its velocity vector can change in two possible ways:
  - The magnitude of the velocity (speed) can change, or
  - The direction of the velocity can change

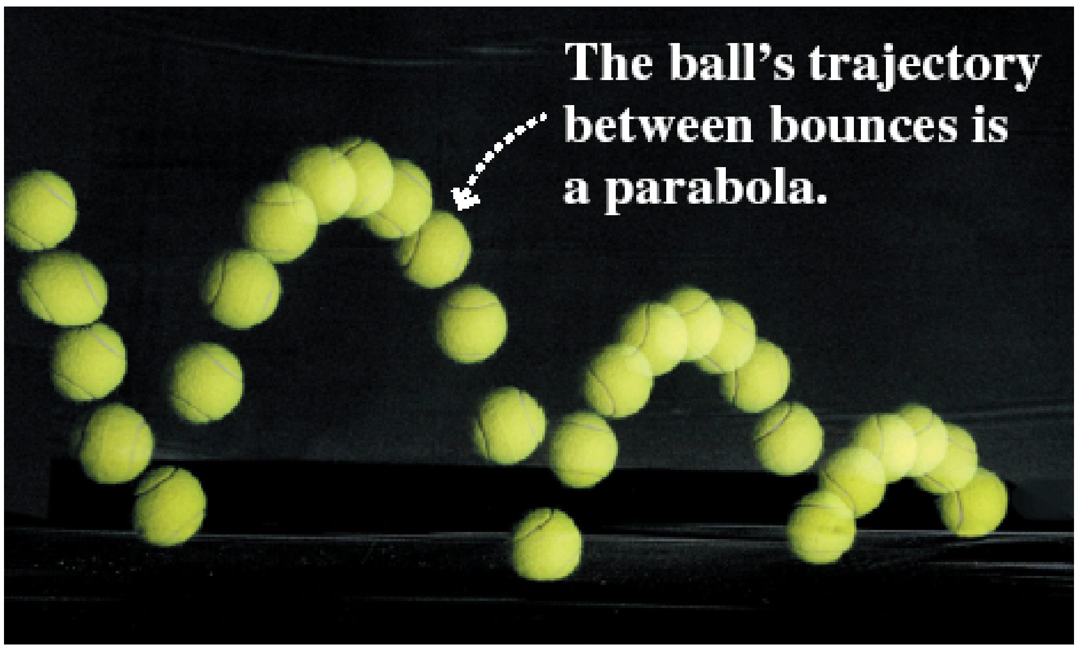
Example: Through the Valley

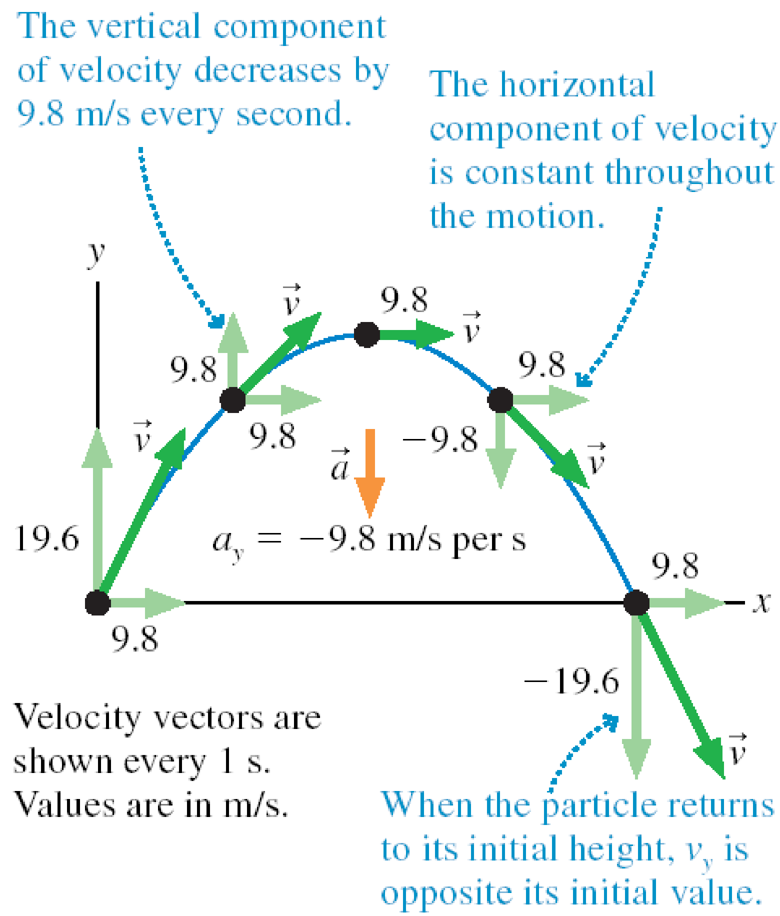
A ball rolls down a long hill, through the valley, and back up the other side.

Draw a complete motion diagram of the ball.



Projectile Motion





Projectile Motion, cont

- Projectile motion is made up of two independent motions:
  - uniform motion in the horizontal direction,
  - free-fall motion in the vertical direction
- The kinematic equations that describe these two motions are

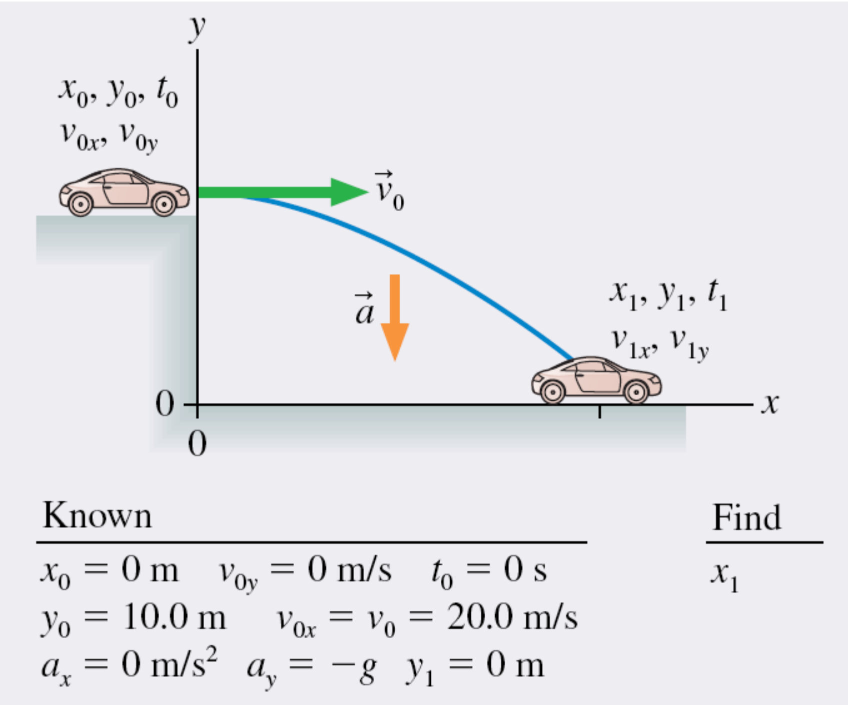
$$x_f = x_i + v_{ix}\,\Delta t \qquad y_f = y_i + v_{iy}\,\Delta t - \frac{1}{2}g(\Delta t)^2$$

$$v_{fx} = v_{ix} = \text{constant} \qquad v_{fy} = v_{iy} - g\,\Delta t$$

Example: A Stunt Man

A stunt man drives a car off a 10.0-m high cliff at a speed of 20.0 m/s.

How far does the car land from the base of the cliff?



- Although the horizontal and vertical motions are independent, they are connected through time *t*

Example: A Stunt Man, cont

- The kinematic equations are:

$$x_1 = x_0 + v_{0x}(t_1 - t_0) = v_0 t_1$$

$$y_1 = 0 = y_0 + v_{0y}(t_1 - t_0) - \tfrac{1}{2}g(t_1 - t_0)^2 = y_0 - \tfrac{1}{2}g t_1^2$$

- We can use the vertical equation to determine the time t_1 needed to fall distance y_0:

$$t_1 = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2(10.0\text{ m})}{9.80\text{ m/s}^2}} = 1.43\text{ s}$$

- We then insert this t_1 to find the distance traveled:

$$x_1 = v_0 t_1 = (20.0\text{ m/s})(1.43\text{ s}) = 28.6\text{ m}$$

Problem Solving Strategy

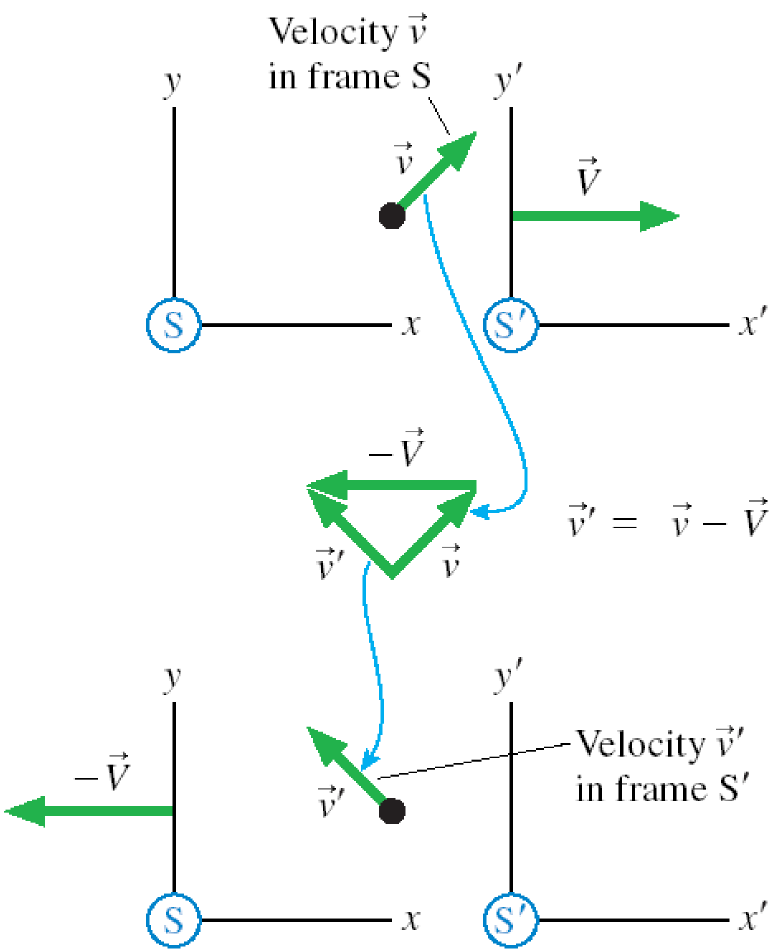
- *Visualize:* Use a pictorial representation
  - With the *x*-axis horizontal and the *y*-axis vertical
  - Define symbols, identify what needs to be found
- *Solve:* Kinematic equations are:

$$x_f = x_i + v_{ix}\,\Delta t \qquad y_f = y_i + v_{iy}\,\Delta t - \frac{1}{2}g(\Delta t)^2$$

$$v_{fx} = v_{ix} = \text{constant} \qquad v_{fy} = v_{iy} - g\,\Delta t$$

  - Acceleration is known: $a_x = 0$, $a_y = -g$
  - Find Δt from one component, and use the value for the other component

Relative Motion



- An object's velocity in one frame can be transformed into another frame:

$$\vec{v} = \vec{v}' + \vec{V} \quad \text{or} \quad \vec{v}' = \vec{v} - \vec{V}$$

(Galilean transformation of velocity)

- In terms of components:

$$v_x = v'_x + V_x \qquad v'_x = v_x - V$$
$$\text{or}$$
$$v_y = v'_y + V_y \qquad v'_y = v_y - V_y$$

Example: A Speeding Bullet

The police are chasing a bank robber. While driving at 50 m/s, they fire a bullet to shoot out a tire of his car.

What is the bullet's speed as measured by a TV camera crew parked beside the road?

Let the Earth be frame S, and a frame attached to the police car be S'. S' moves relative to S with V_x = 50 m/s.

The gun is in S'. So the bullet travels in S' with v'_x= 300 m/s.

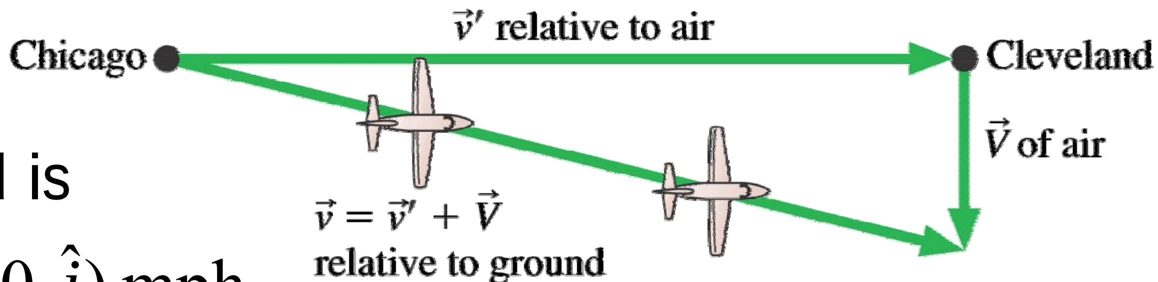
The camera crew is in S. The bullet's velocity in S is:

$$v_x = v'_x + V_x = 300 \text{ m/s} + 50 \text{ m/s} = 350 \text{ m/s}$$

Example: Flying to Cleveland I

Cleveland is 300 miles E of Chicago. A plane leaves Chicago flying due E at 500 mph. The pilot forgot to check the weather and doesn't know that the wind is blowing to the S at 50 mph.

(a) What is the plane's ground speed? (b) Where is the plane 0.60 h later, when the pilot expects to land in Cleveland?



(a) The plane's velocity relative to the ground is

$$\vec{v} = \vec{v}' + \vec{V} = (500\,\hat{i} - 50\,\hat{j})\text{ mph}$$

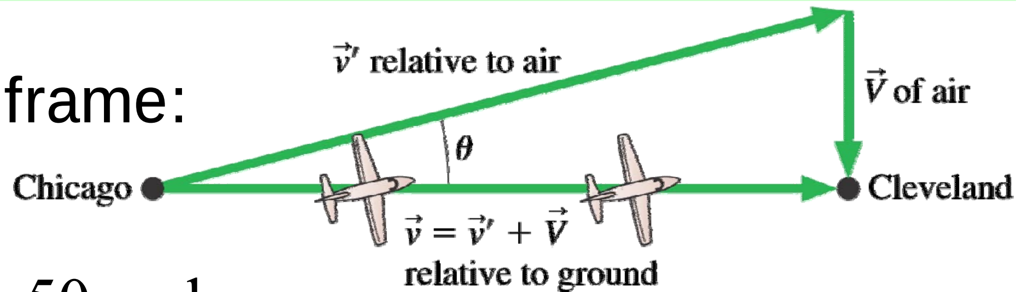
The plane's ground speed is $v = \sqrt{v_x^2 + v_y^2} = 502\text{ mph}$

(b) After flying 0.6 h, the plane is at $(300\text{ mi}, -30\text{ mi})$

Example: Flying to Cleveland II

A wiser pilot flying from Chicago to Cleveland on the same day plots a course that will take her directly to Cleveland.

(a) In which direction does she fly the plane? (b) How long does it take to reach Cleveland?



(a) Plane's velocity in earth's frame:

$$v_x = v'_x + V_x = (500\text{ mph})\cos\theta$$

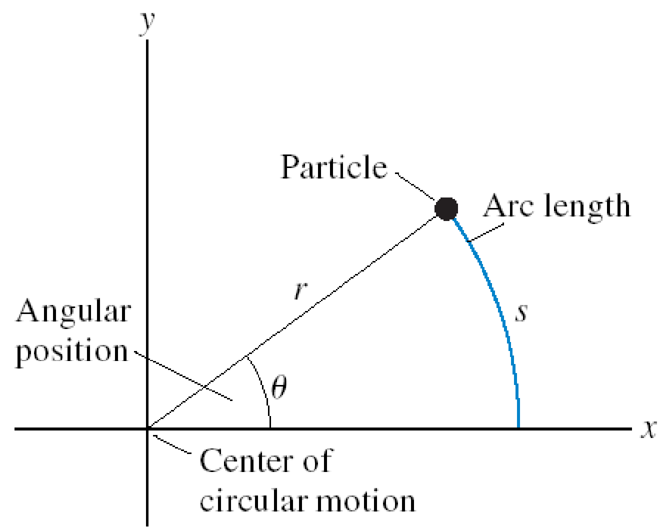
$$v_y = v'_y + V_y = (500\text{ mph})\sin\theta - 50\text{ mph}$$

From $v_y = 0$, proper heading is $\theta = \sin^{-1}\left(\dfrac{50\text{ mph}}{500\text{ mph}}\right) = 5.74°$

(b) The time it takes: $t = \dfrac{300\text{ mi}}{(500\text{ mi})\cos 5.74°} = 0.604\text{ h}$

(14 s longer)

Circular Motion



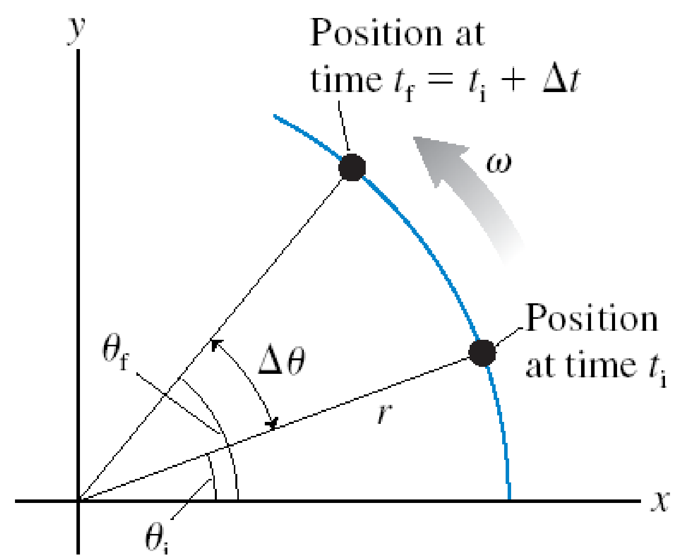
- Circular motion can be described in terms of the angular position θ
  - Usually we will measure θ in radians:

$$\theta\,(\text{in rad}) = s / r$$

  - 2π rad = 360°

- Angular velocity is defined as the rate of change of θ

$$\omega_{ave} = \frac{\Delta\theta}{\Delta t} \quad \text{or} \quad \omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$



Circular Motion, cont

- We will also define an angular acceleration, α

$$\alpha_{ave} = \frac{\Delta\omega}{\Delta t} \text{ or } \alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

- For uniform circular motion (UCM), ω is constant and hence α = 0.
  - This does not mean a = 0.

Uniform Circular Motion

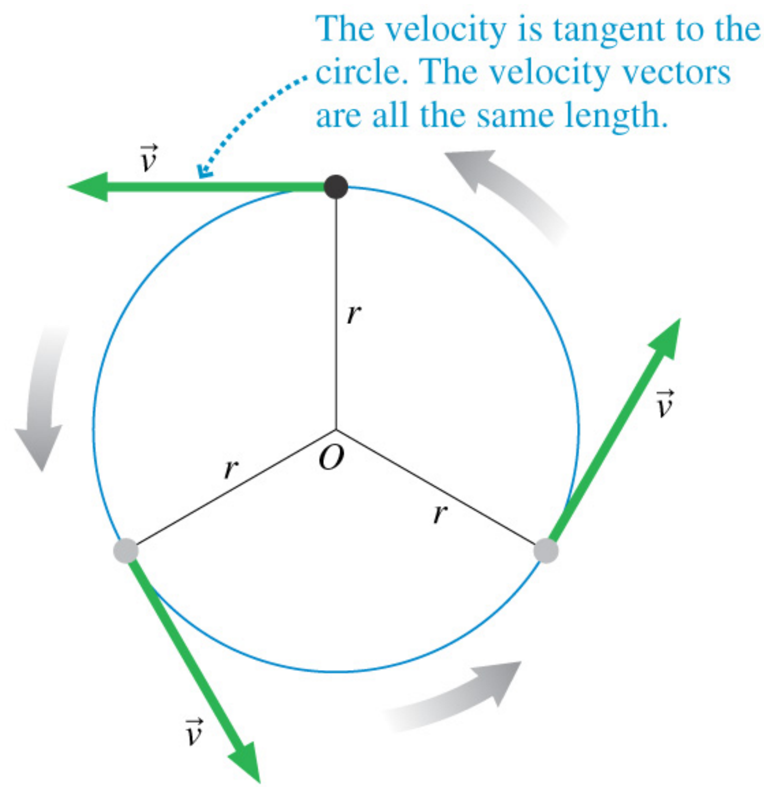
- UCM is a constant speed motion in a circle

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} \quad (T : \text{the period})$$

- The speed of a point on the circle is

$$v = \frac{\text{circumference}}{\text{period}} = \frac{2\pi r}{T} = \omega r$$

- The velocity vector is tangential to the circle



Example: Roulette Wheel

A small roulette ball rolls ccw around the inside of a 30-cm diameter roulette wheel. The ball completes 2.0 rev in 1.20 s.

(a) What is the ball's angular velocity?

(b) What is the ball's position at t = 2.0 s. Assume $\theta_i = 0$.

(a) Period of the ball's motion is T = 0.60 s.

Angular velocity is positive and $\omega = \dfrac{2\pi\ \text{rad}}{0.60\ \text{s}} = 10.47\ \text{rad/s}$

(b) Ball's position at Δt = 2.0 s is

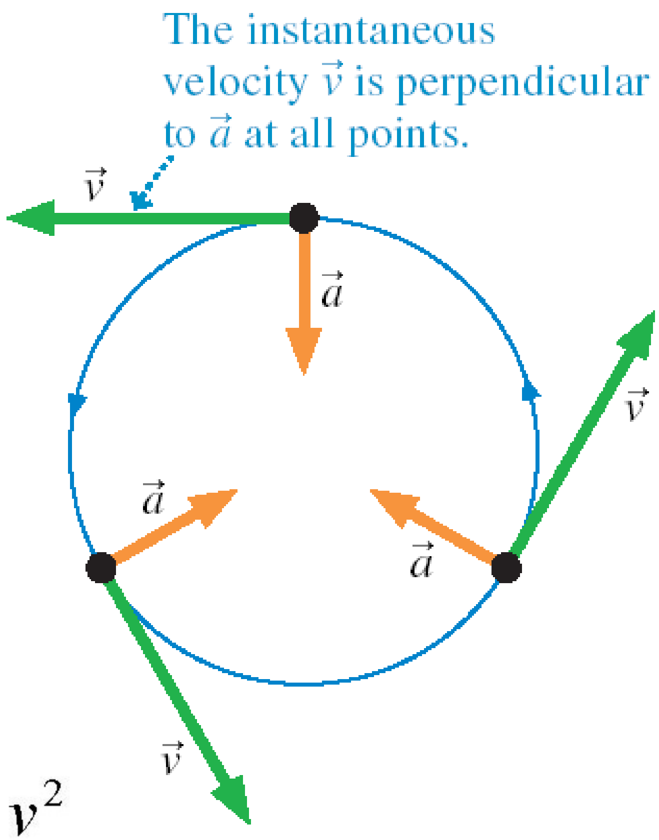
$$\theta_f = 0\ \text{rad} + (10.47\ \text{rad/s})(2.0\ \text{s}) = 20.94\ \text{rad} = 3.333 \times 2\pi\ \text{rad}$$

We subtract an integer number of 2π rad:

$$\theta_f' = 20.94\ \text{rad} - 3 \times 2\pi\ \text{rad} = 2.09\ \text{rad} = 120°$$

Uniform Circular Motion, cont

- An acceleration exists since the *direction* of the velocity vector is changing
  - The acceleration is always perpendicular to the path and points to the center
- This acceleration is called the *centripetal acceleration*
  - The magnitude of the centripetal acceleration is $a_C = \frac{v^2}{r}$

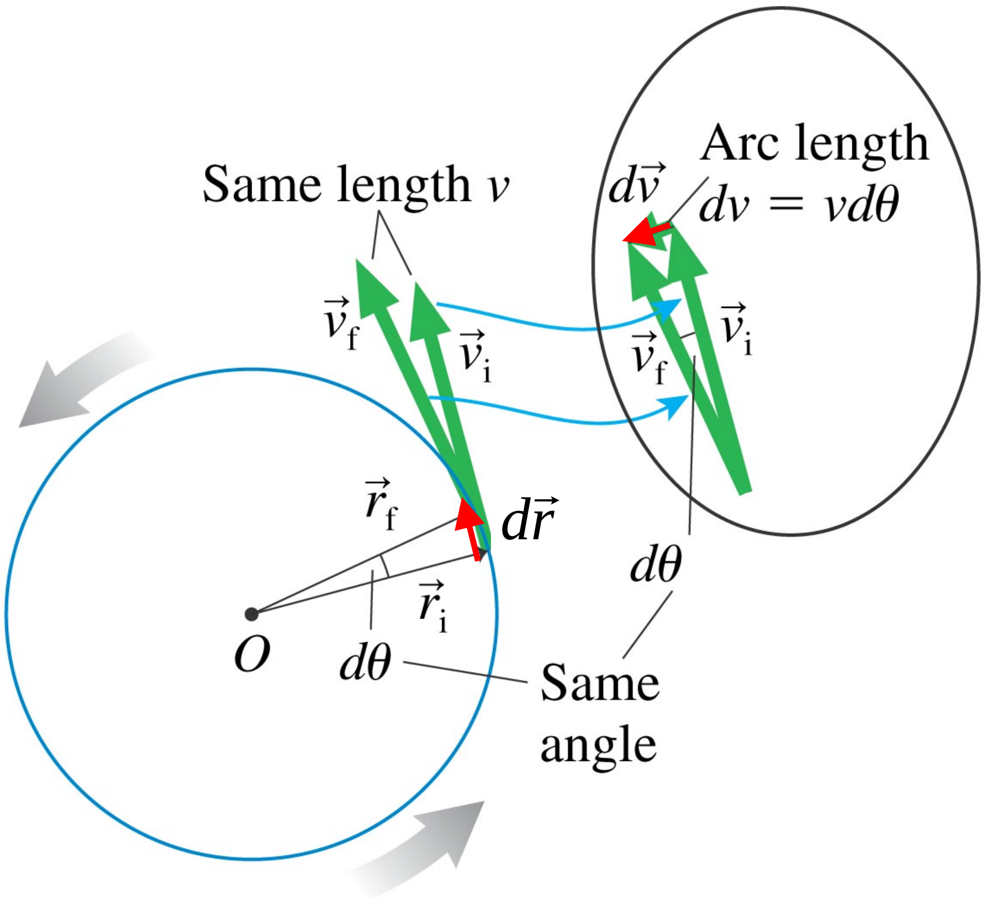


Centripetal Acceleration

- From geometry,

$$\frac{dv}{v} = \frac{dr}{r}, \quad dv = \frac{v}{r} dr$$

$$a_C = \frac{dv}{dt} = \frac{v}{r}\frac{dr}{dt} = \frac{v^2}{r}$$



- The direction is toward the center:

$$\vec{a}_C = -\frac{v^2}{r}\hat{r}$$

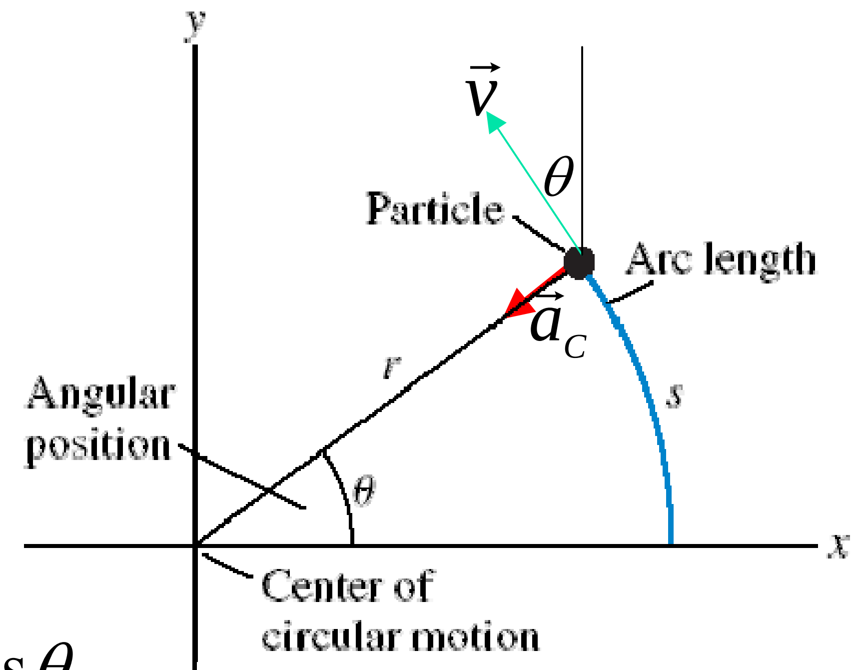
Centripetal Acceleration, cont

- From geometry,

$$v_x = -v\sin\theta$$

$$v_y = v\cos\theta$$

where v is a constant.



- Using calculus,

$$a_x = \frac{dv_x}{dt} = -v\cos\theta\frac{d\theta}{dt} = -\frac{v^2}{r}\cos\theta,$$

$$a_y = \frac{dv_y}{dt} = -v\sin\theta\frac{d\theta}{dt} = -\frac{v^2}{r}\sin\theta$$

$$\vec{a}_C = -\frac{v^2}{r}\hat{r}$$

Example: Ferris Wheel

A carnival Ferris wheel has a radius of 9.0 m and rotates 4.0 times per minute.

What acceleration do the riders experience?

Period is T = ¼ min = 15 s.

A rider's speed is $$v = \frac{2\pi r}{T} = \frac{2\pi(9.0\text{ m})}{15\text{ s}} = 3.77\text{ m/s}$$

Therefore, the centrifugal acceleration is

$$a_C = \frac{v^2}{r} = \frac{(3.77\text{ m/s})^2}{9.0\text{ m}} = 1.6\text{ m/s}^2$$

Non-uniform Circular Motion

- The magnitude of the velocity could also be changing
  - If the speed is changing, this is *not* a Uniform Circular motion
- In this case, there would be a *tangential acceleration* as well
  - The magnitude of this acceleration is $a_t = \frac{d|\mathbf{v}|}{dt}$

Total Acceleration

- Tangential acceleration:

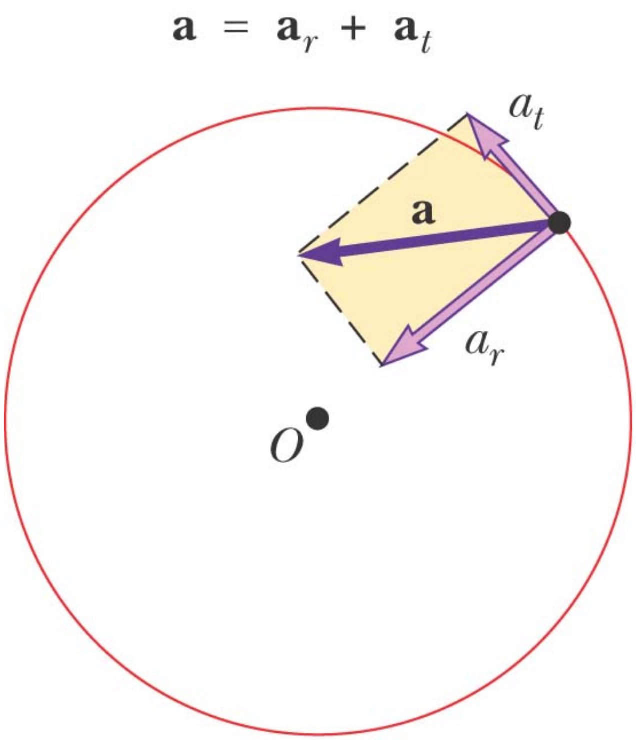
$$a_t = \frac{d\left|\mathbf{v}\right|}{dt}$$

- Radial acceleration:

$$a_r = -a_C = -\frac{v^2}{r}$$

- The total acceleration:

$$a = \sqrt{a_r^2 + a_t^2}$$

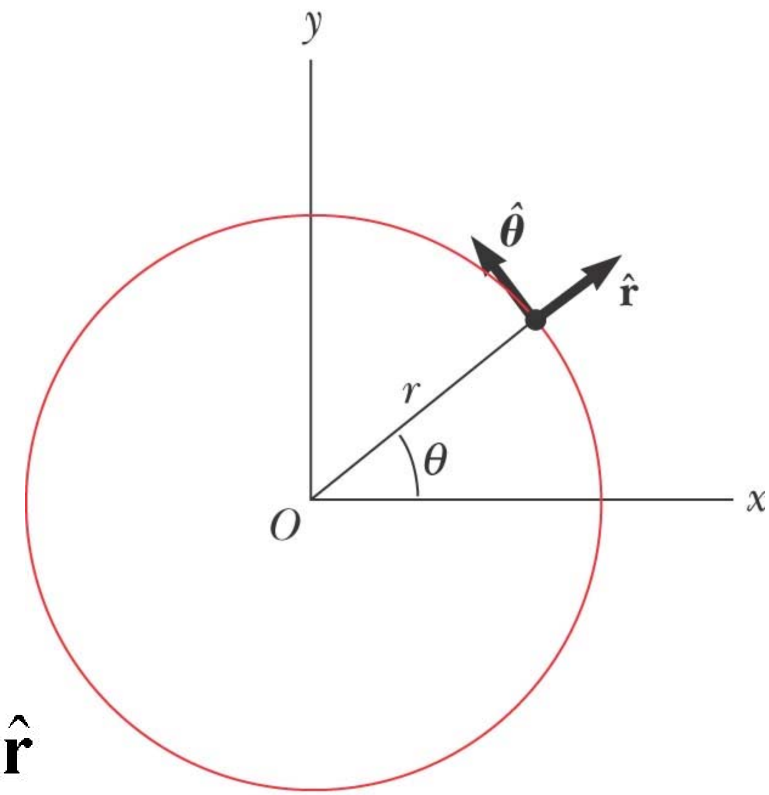


Total Acceleration, cont

- Define unit vectors:

  $\hat{r}$: radially outward

  $\hat{\theta}$: tangent in the direction of increasing θ



- Total acceleration is

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r = \frac{d|\mathbf{v}|}{dt}\hat{\theta} - \frac{v^2}{r}\hat{\mathbf{r}}$$

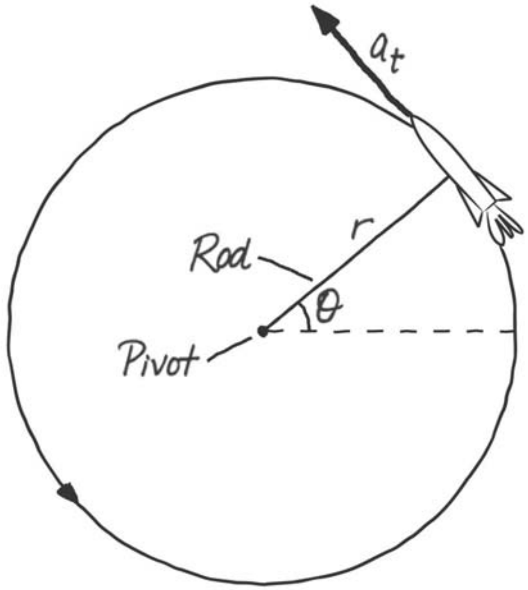
  - The sign of a_t must be provided manually

Example: Circular Rocket Motion

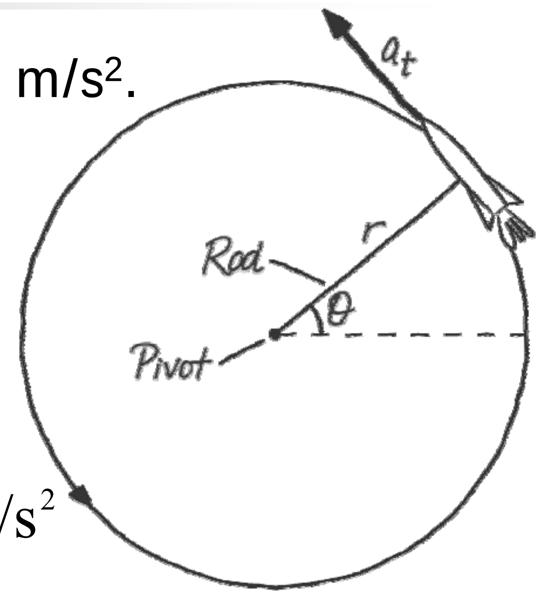
A model rocket is attached to the end of a 2.0-m-long rigid rod. The other end of the rod rotates on a frictionless pivot, causing the rocket to move in a horizontal circle. The rocket accelerates at 1.0 m/s^2 for 10 s, starting from rest, then runs out of fuel.

(a) What is the magnitude of **a** at t = 2.0 s?

(b) What is the rocket's angular velocity, in rpm, when it runs out of the fuel?



Circular Rocket Motion, cont



(a) The rocket creates a tangential acceleration a_t = 1.0 m/s^2.

At t = 2.0 s, $v = v_0 + a_t\Delta t$ = 2.0 m/s. The rocket then acquires a radial acceleration

$$a_t = \frac{v^2}{r} = \frac{(2.0\ \mathrm{m/s})^2}{2.0\ \mathrm{m}} = 2.0\ \mathrm{m/s^2},$$

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{(1.0\ \mathrm{m/s^2})^2 + (2.0\ \mathrm{m/s^2})^2} = 2.2\ \mathrm{m/s^2}$$

(b) The tangential velocity after 10 s is $v = v_0 + a_t\Delta t$ = 10.0 m/s.

Thus the angular velocity is $\omega = \frac{v}{r} = \frac{10.0\ \mathrm{m/s}}{2.0\ \mathrm{m}} = 5.0\ \mathrm{rad/s}$

Converting to rpm, $\omega = \frac{5.0\ \mathrm{rad}}{1\ \mathrm{s}} \times \frac{1\ \mathrm{rev}}{2\pi\ \mathrm{rad}} \times \frac{60\ \mathrm{s}}{1\ \mathrm{min}} = 48\ \mathrm{rpm}$

Linear Versus Angular Motion

	Linear Motion	Angular Motion
Displacement	Δs	$\Delta\theta$
Velocity	$v_{avg} = \frac{\Delta s}{\Delta t}, \ v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$	$\omega_{avg} = \frac{\Delta\theta}{\Delta t}, \ \omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$
Acceleration	$a_{avg} = \frac{\Delta v}{\Delta t}, \ a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$	$\alpha_{avg} = \frac{\Delta\omega}{\Delta t}, \ \alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$
Kinematic equations	$v = v_0 + a\Delta t$ $\Delta s = v_0\Delta t + \frac{1}{2}a(\Delta t)^2$ $v^2 = v_0^2 + 2a\Delta s$	$\omega = \omega_0 + \alpha\Delta t$ $\Delta\theta = \omega_0\Delta t + \frac{1}{2}\alpha(\Delta t)^2$ $\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$

General Principles

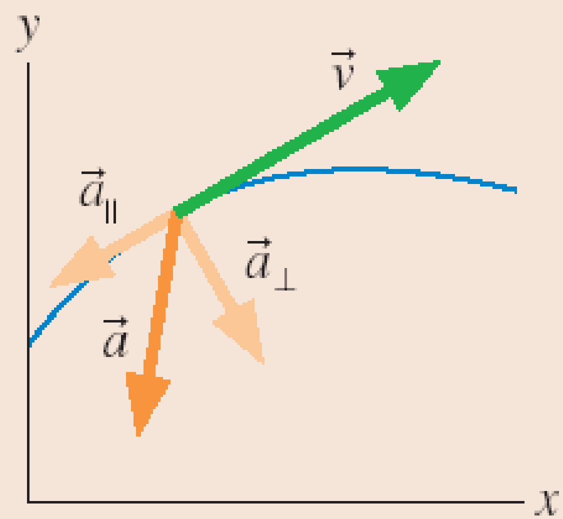
The instantaneous velocity

$$\vec{v} = d\vec{r}/dt$$

is a vector tangent to the trajectory.

The instantaneous acceleration is

$$\vec{a} = d\vec{v}/dt$$



$\vec{a}_{\parallel}$, the component of $\vec{a}$ parallel to $\vec{v}$, is responsible for change of *speed*. $\vec{a}_{\perp}$, the component of $\vec{a}$ perpendicular to $\vec{v}$, is responsible for change of *direction*.

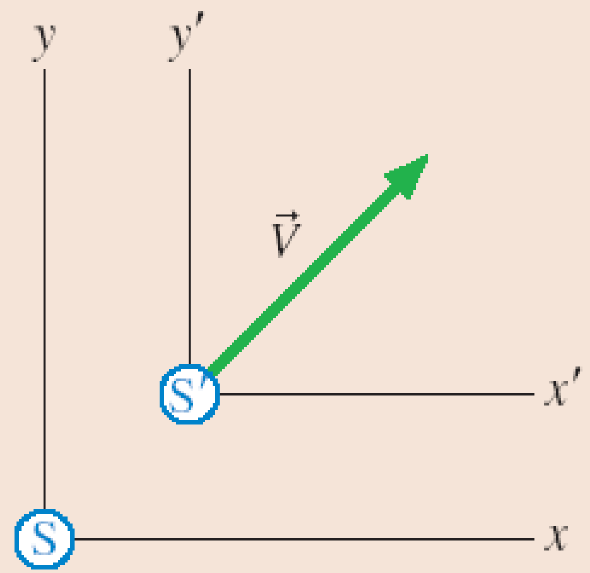
General Principles

Relative motion

Inertial reference frames move relative to each other with constant velocity $\vec{V}$. Measurements of position and velocity measured in frame S are related to measurements in frame S′ by the Galilean transformations:

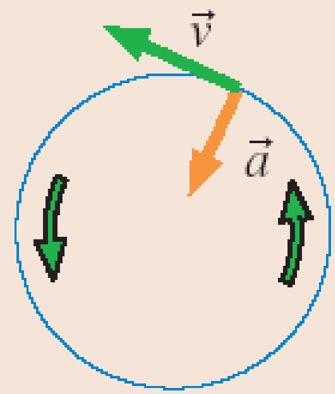
$$x' = x - V_x t \qquad v'_x = v_x - V_x$$

$$y' = y - V_y t \qquad v'_y = v_y - V_y$$



Important Concepts

Uniform Circular Motion



Angular velocity $\omega = d\theta/dt$.

v_t and ω are constant:

$$v_t = \omega r$$

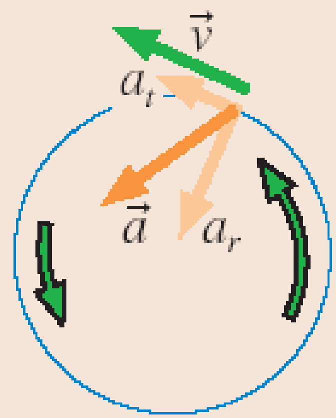
The centripetal acceleration points toward the center of the circle:

$$a = \frac{v^2}{r} = \omega^2 r$$

It changes the particle's direction but not its speed.

Important Concepts

Nonuniform Circular Motion



Angular acceleration $\alpha = d\omega/dt$.
The radial acceleration

$$a_r = \frac{v^2}{r} = \omega^2 r$$

changes the particle's direction. The tangential component

$$a_t = \alpha r$$

changes the particle's speed.

Applications

Kinematics in two dimensions

If $\vec{a}$ is constant, then the x- and y-components of motion are independent of each other.

$$x_f = x_i + v_{ix}\Delta t + \tfrac{1}{2}a_x(\Delta t)^2$$

$$y_f = y_i + v_{iy}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

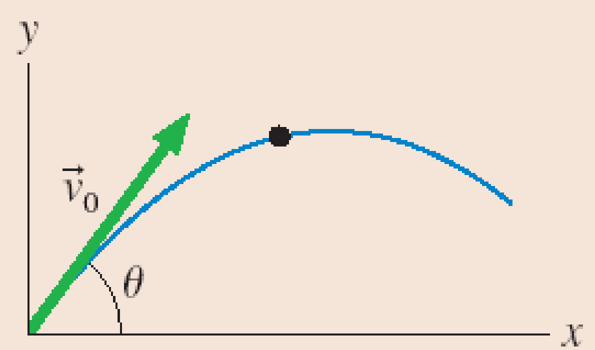
$$v_{fx} = v_{ix} + a_x\Delta t$$

$$v_{fy} = v_{iy} + a_y\Delta t$$

Applications

Projectile motion occurs if the object moves under the influence of only gravity. The motion is a parabola.

- Uniform motion in the horizontal direction with $v_{0x} = v_0 \cos\theta$.
- Free-fall motion in the vertical direction with $a_y = -g$ and $v_{0y} = v_0 \sin\theta$.
- The x and y kinematic equations have the *same* value for Δt.

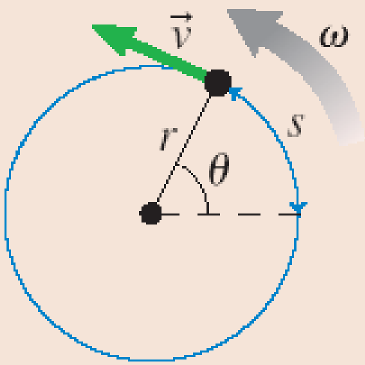


Applications

Circular motion kinematics

Period $T = \dfrac{2\pi r}{v} = \dfrac{2\pi}{\omega}$

Angular position $\theta = \dfrac{s}{r}$



$$\omega_f = \omega_i + \alpha\,\Delta t$$

$$\theta_f = \theta_i + \omega_i\,\Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\,\Delta\theta$$

Applications

Angle, angular velocity, and angular acceleration are related graphically.

- The angular velocity is the slope of the angular position graph.
- The angular acceleration is the slope of the angular velocity graph.