Physics for Scientists and Engineers

Chapter 3

Vectors and Coordinate Systems

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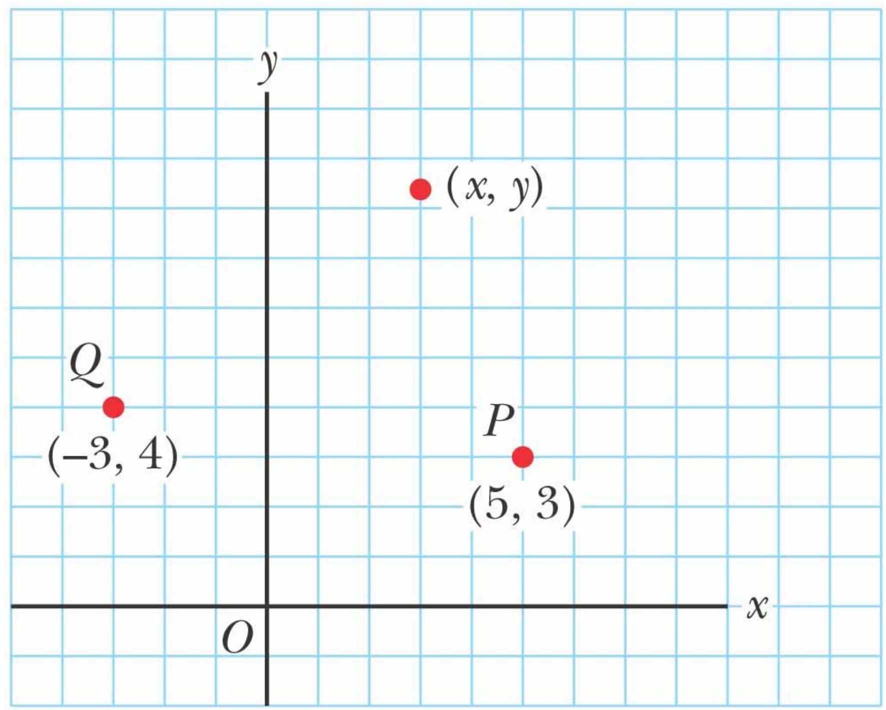
Coordinate Systems

- Used to describe the position of a point in space
- Coordinate system consists of
  - a fixed reference point called the origin
  - specific axes with scales and labels
  - instructions on how to label a point relative to the origin and the axes

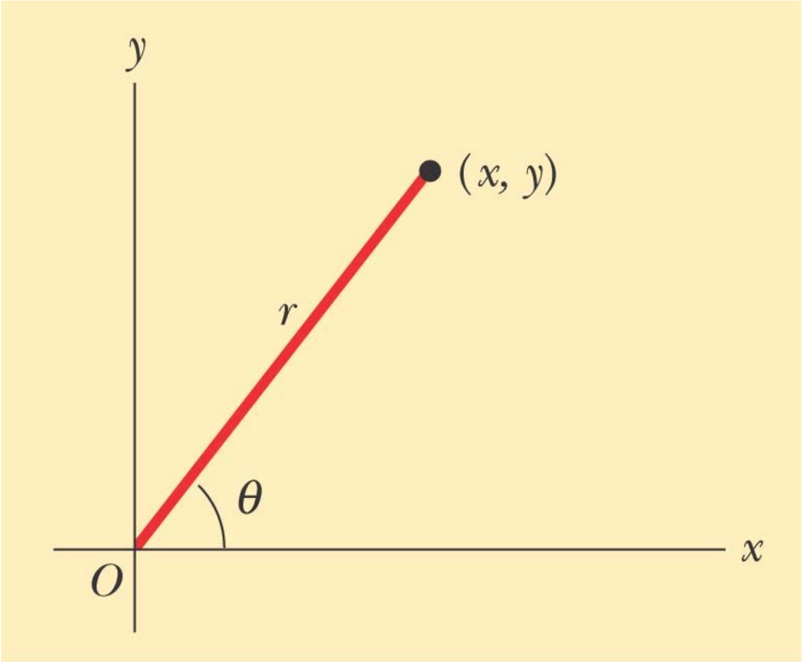
Cartesian Coordinate System

- Also called rectangular coordinate system
- *x*- and *y*- axes intersect at the origin
- Points are labeled (*x*, *y*)



Polar Coordinate System

- Origin and reference line are noted
- Point is at distance r from the origin in the direction of angle θ, ccw from reference line (positive x axis)
- Points are labeled as (r, θ)



Coordinate Transformations

- Polar to Cartesian coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

- Cartesian to polar coordinates:

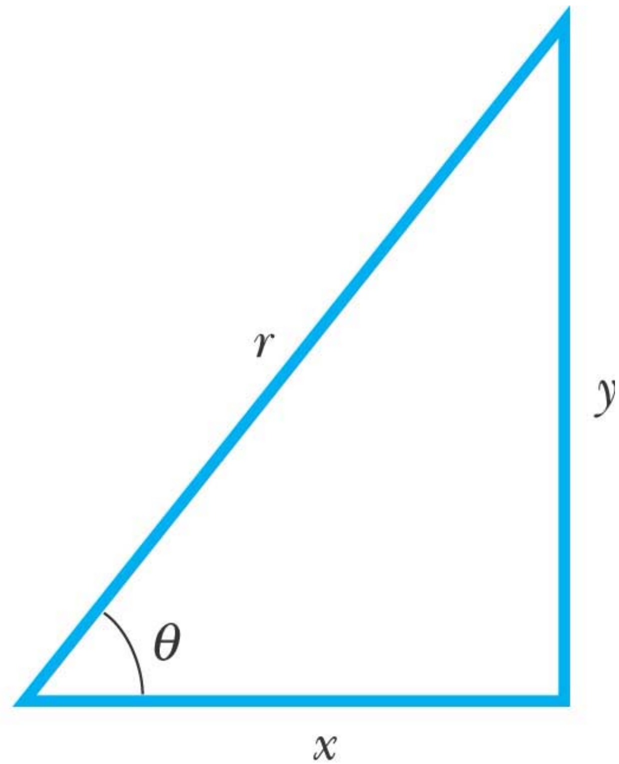
$$\tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{r}$$

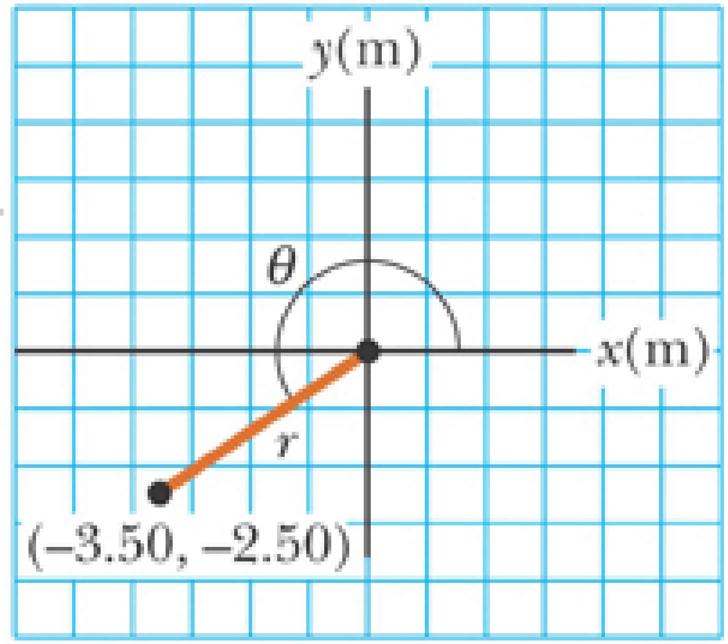
$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$



Example 1

The Cartesian coordinates of a point in the *xy* plane are (*x, y*) = (-3.50, -2.50) m, as shown in the figure. Find the polar coordinates of this point.



$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50\text{ m})^2 + (-2.50\text{ m})^2} = 4.30\text{ m}$$

$$\tan\theta = \frac{y}{x} = \frac{-2.50\text{ m}}{-3.50\text{ m}} = 0.714$$

$$\theta = 216°$$

Note: For most calculators,
$\tan^{-1} 0.714 = 36°$

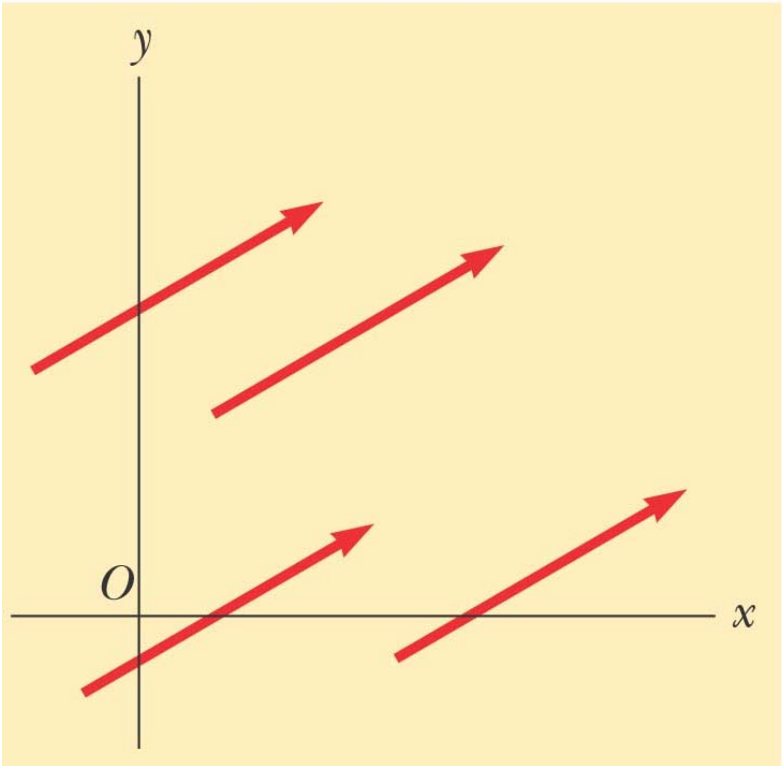
We need to add 180°:
$\theta = 180° + 36° = 216°$

Vector Notation

- When handwritten, use an arrow: $\vec{A}$
- When printed, will be in bold print: **A**
- The magnitude of a vector is indicated as: *A* or |**A**|
  - The magnitude of a vector is always a positive number
  - The magnitude of the vector has physical units: m/s, m/s^2, etc

Equality of Two Vectors

- Two vectors are *equal* if they have the same magnitude and the same direction
- **A** = **B** if $A = B$ and they point along parallel lines
  - All of the vectors shown are equal

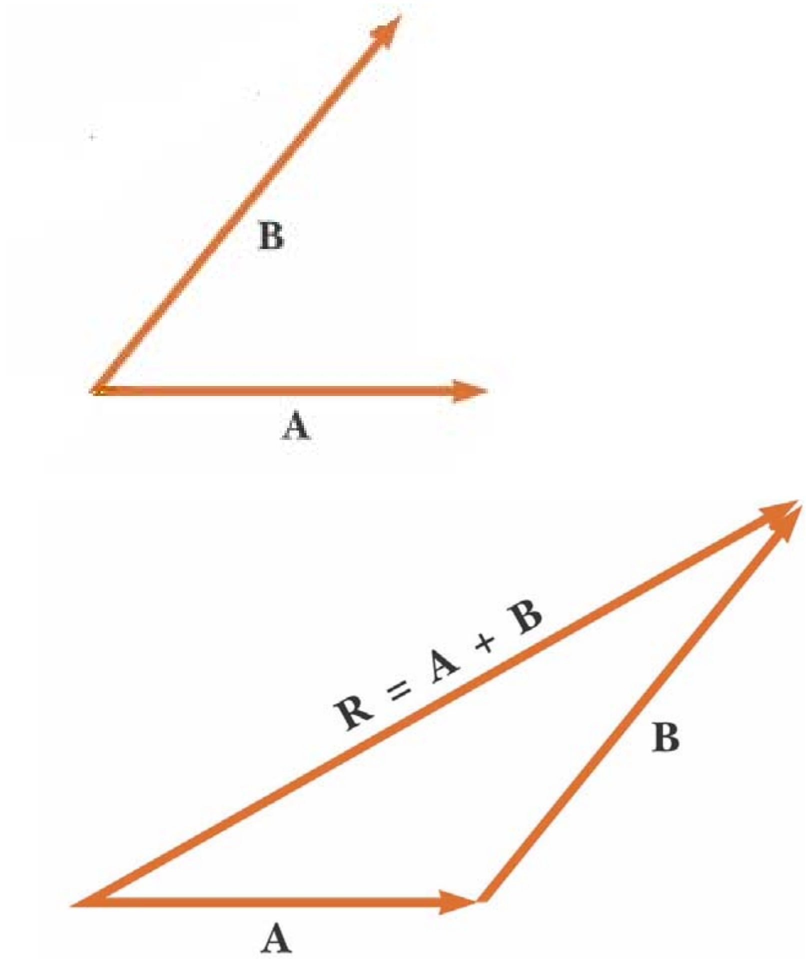


Adding Vectors

- When adding vectors, their directions must be taken into account
- All the vectors must be of the same type of quantity with the same units
- Graphical Methods
  - Use scale drawings
- Algebraic Methods
  - More convenient

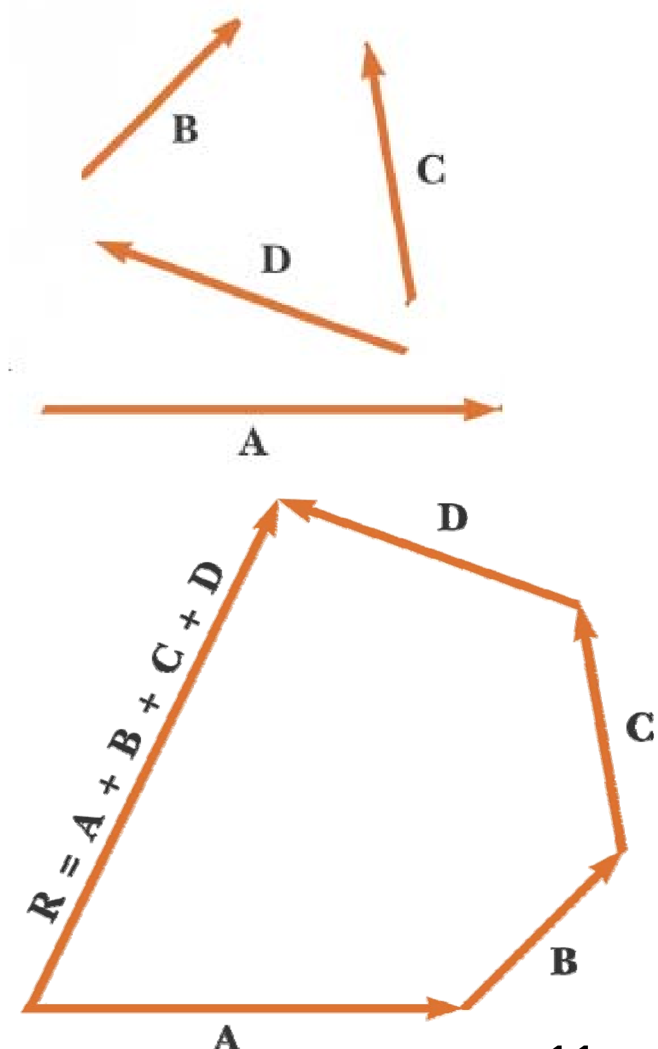
Adding Vectors Graphically

- Choose a scale
- Draw the vectors “tip-to-tail”
- The resultant is drawn from the origin of the first to the end of the last vector



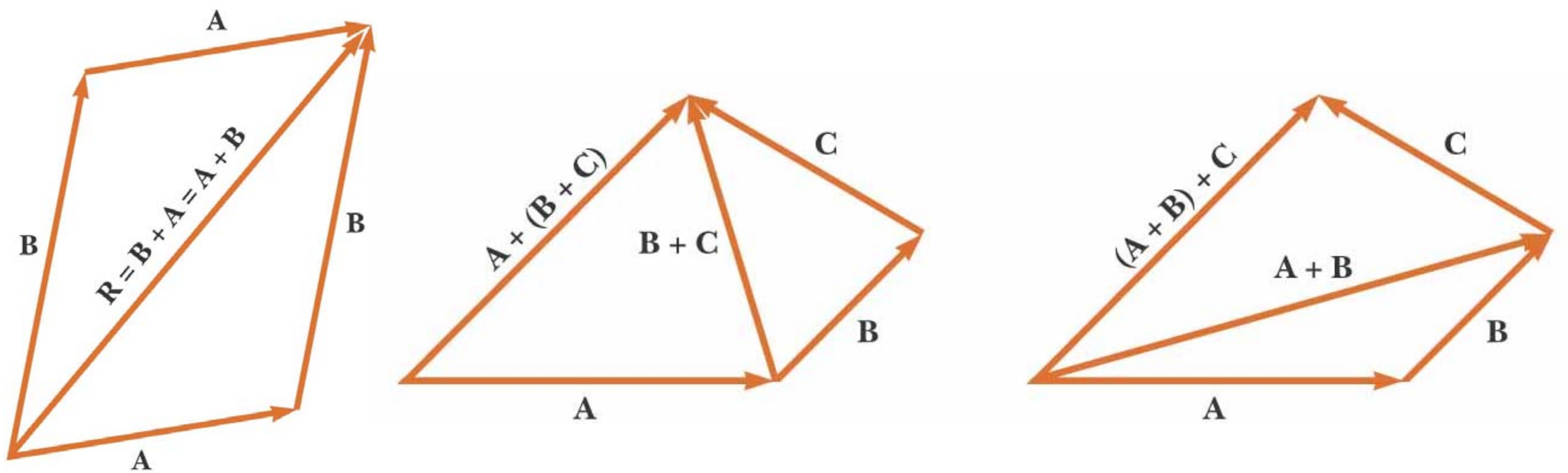
Adding Vectors Graphically, cont

- When you have many vectors, just keep repeating the process until all are included
- The resultant is still drawn from the origin of the first vector to the end of the last vector



Adding Vectors, Rules

- The sum is independent of the order of the addition
  - *Commutative law:* **A** + **B** = **B** + **A**
- The sum is independent of the way in which the individual vectors are grouped
  - *Associative law:* **A** + (**B** + **C**) = (**A** + **B**) + **C**

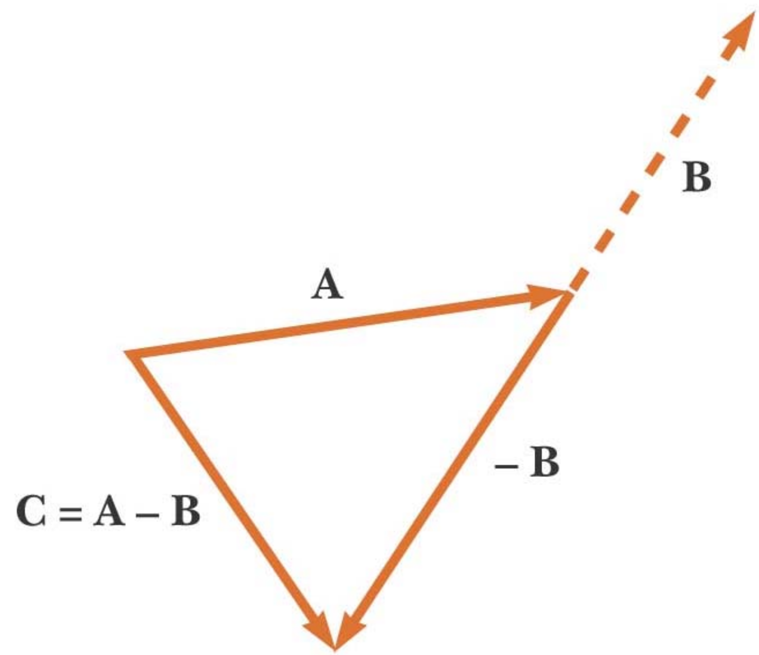


Negative of a Vector

- The negative of a vector is defined as the vector that, when added to the original vector, gives a resultant of zero
  - Represented as –**A**
  - **A** + (–**A**) = 0
- The negative of the vector will have the same magnitude, but point in the opposite direction

Subtracting Vectors

- Special case of vector addition
- **A** – **B** = **A** + (–**B**)
- Continue with standard vector addition procedure

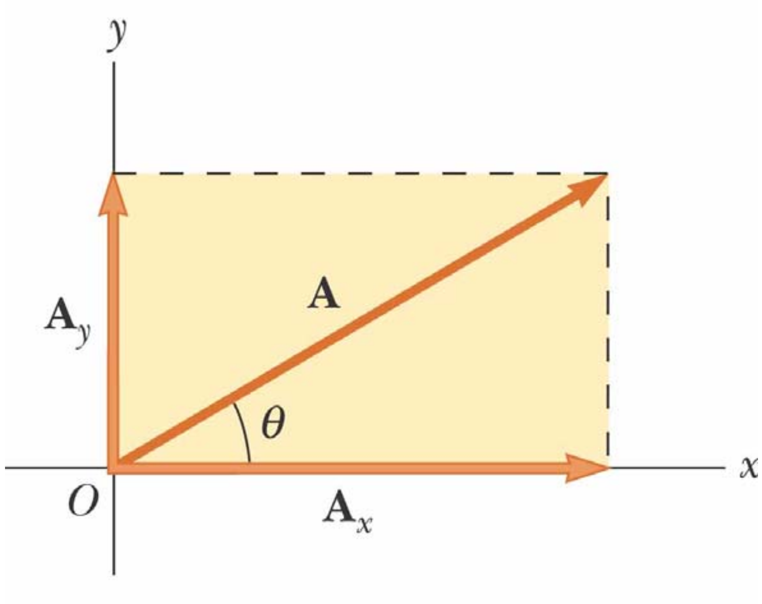


Multiplying a Vector by a Scalar

- The result of the multiplication or division of a vector by a scalar is a vector
- The magnitude of the vector is multiplied or divided by the scalar
- The direction of the resultant vector depends on the sign of the scalar
  - If scalar > 0, the direction is the same as of the original vector
  - If scalar < 0, the direction is opposite to that of the original vector

Components of a Vector

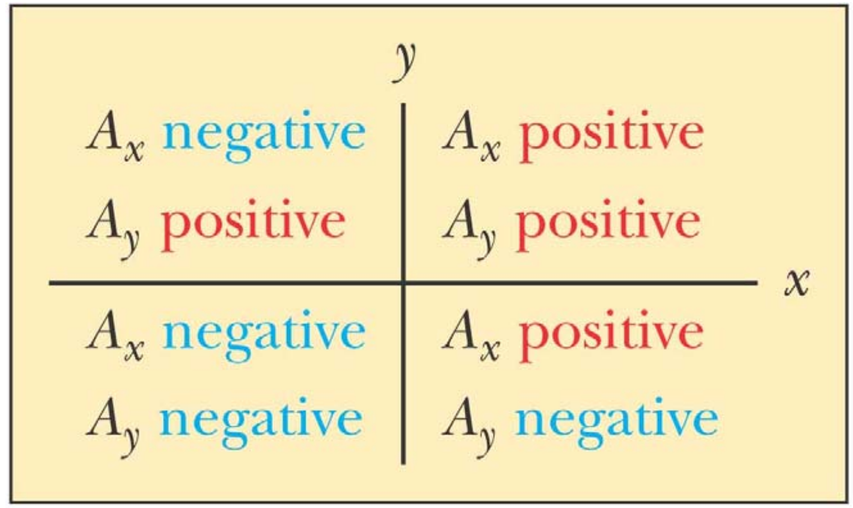
- Components are the projections of a vector along *x*- and *y*-axes
- $\mathbf{A_x}$ and $\mathbf{A_y}$ are the *component vectors* of **A**
  - $\mathbf{A} = \mathbf{A_x} + \mathbf{A_y}$
- A_x and A_y are the "*components*" of **A**
  - $A_x = A\cos\theta,\ A_y = A\sin\theta$



- *Note:* θ must be measured ccw from the positive *x*-axis!

Components of a Vector, cont

- The signs of the components will depend on θ
- The magnitude and direction of **A** can be found from:

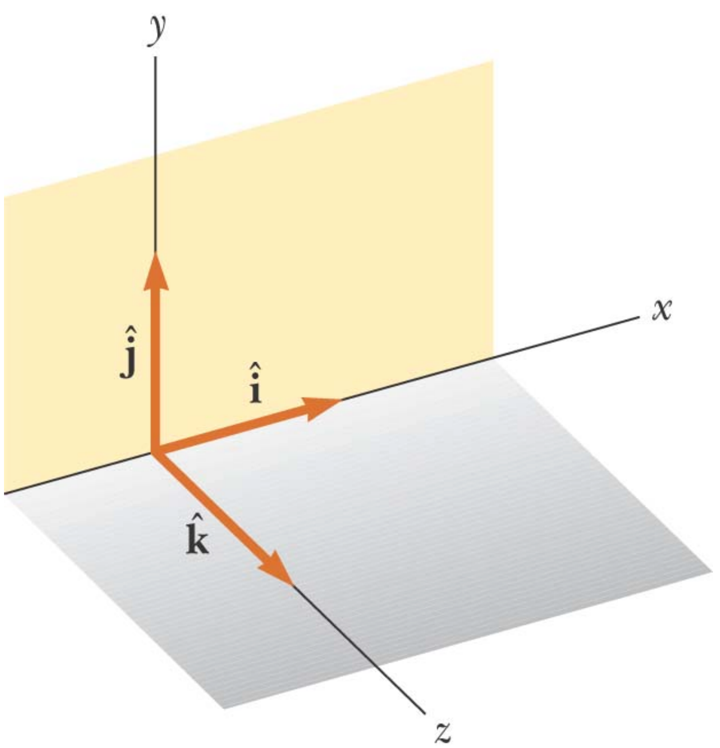


$$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{A_y}{A_x}$$

- The components have the same units as the original vector

Unit Vectors

- A *unit vector* is a dimensionless vector with a magnitude of 1
- Unit vectors are used to specify a direction and has no other significance
- The symbols $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ represent unit vectors
  - They form a set of mutually perpendicular vectors

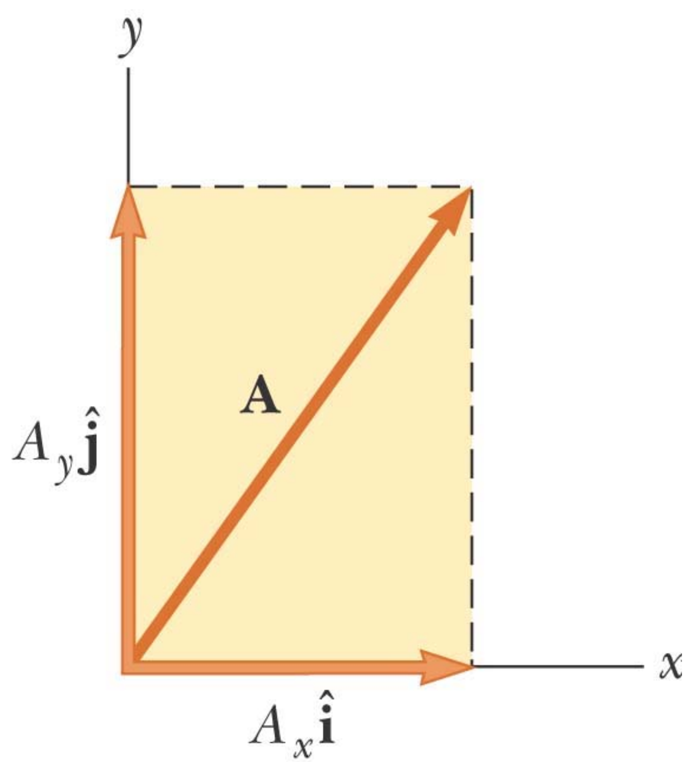


Vectors in Terms of Unit Vectors

- $\mathbf{A_x} = A_x\hat{\mathbf{i}}$ and $\mathbf{A_y} = A_y\hat{\mathbf{j}}$, etc.
- The complete vector can be expressed as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

  - A_x, A_y, A_z can be positive or negative



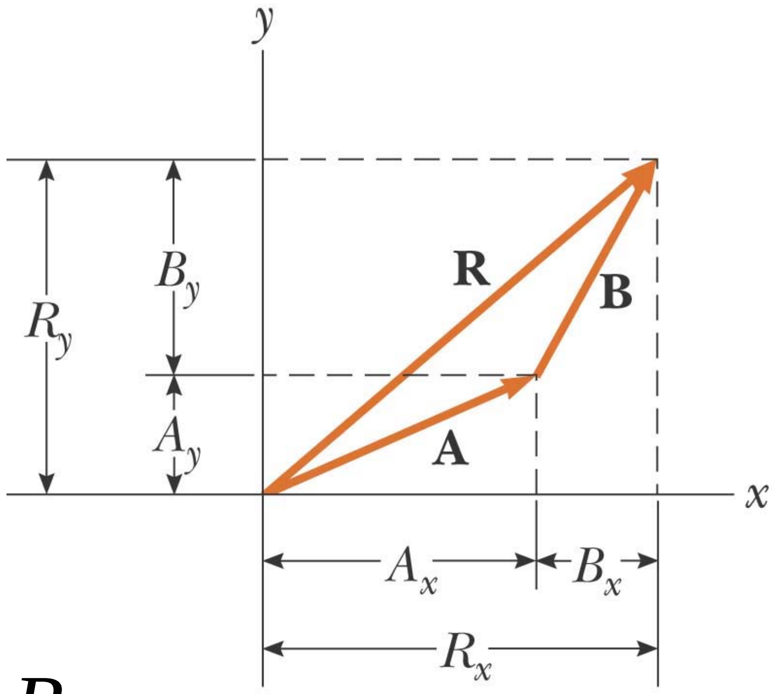
Adding Vectors in Unit Vectors

- Writing **R** = **A** + **B** in unit vectors:

$$R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$$

$$= (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}) + (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}})$$

$$= (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}}$$



- So $R_x = A_x + B_x, R_y = A_y + B_y$
- From R_x and R_y, $R = \sqrt{R_x^2 + R_y^2}$ $\theta = \tan^{-1}\frac{R_y}{R_x}$

Adding Vectors in 3-D

- Writing **R** = **A** + **B** in unit vectors:

$$R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}} + R_z\hat{\mathbf{k}}$$

$$= (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) + (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})$$

$$= (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}}$$

- So $R_x = A_x + B_x$, $R_y = A_y + B_y$, $R_z = A_z + B_z$
- From R_x and R_y,

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \quad \theta_x = \tan^{-1}\frac{R_x}{R}$$

Example 2: Taking a Hike

A hiker begins a trip by first walking 25.0 km SE from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° N of E, at which point she discovers a forest ranger's tower.

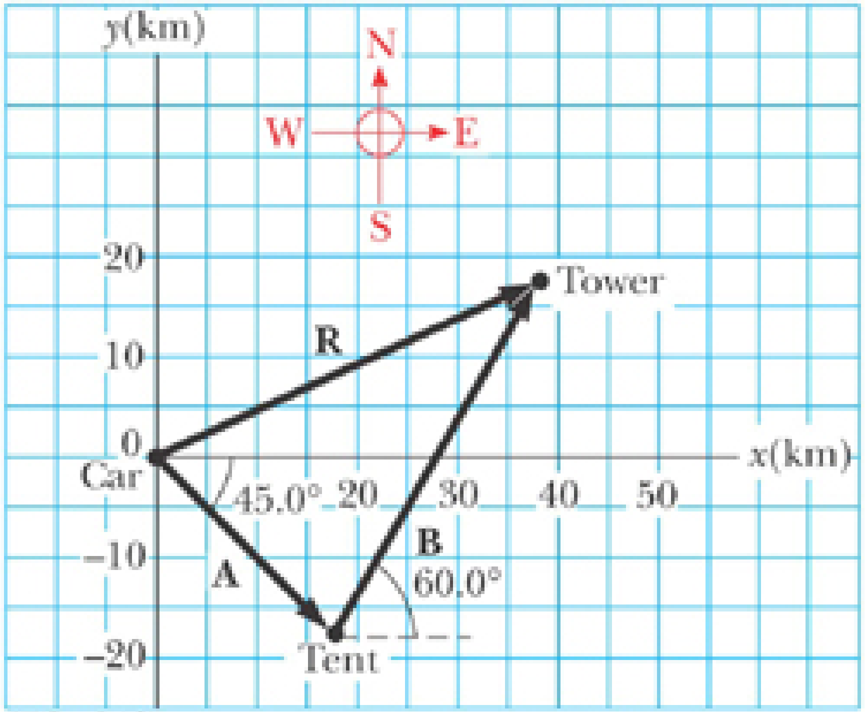
(a) Determine the components of the hiker's displacement for each day.

(b) Determine the components of the hiker's resultant displacement **R** for the trip. Find an expression for **R** in terms of unit vectors.

Example 2, cont

- *Conceptualize:*

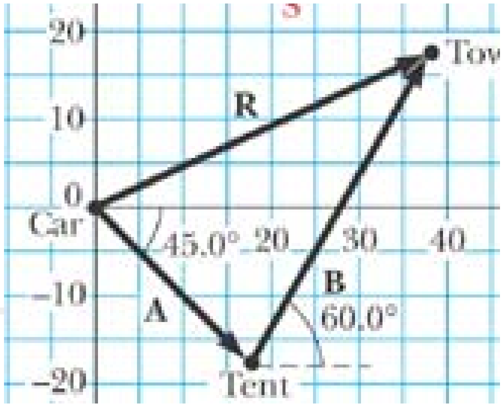
  Draw a sketch. We can use the car as the origin of coordinates and denote the displacement vectors on the first and second days by **A** and **B**, respectively.



- *Categorize:*

  We can now see this problem as an addition of two vectors to find the resultant vector **R**.

Example 2, cont



- *Analyze:*

(a) **A** has a magnitude of 25.0 km and is 45.0° SE

$$A_x = A\cos(-45.0°) = (25.0\text{ km})(0.707) = 17.7\text{ km}$$

$$A_y = A\sin(-45.0°) = (25.0\text{ km})(-0.707) = -17.7\text{ km}$$

B has a magnitude of 40.0 km and is 60.0° N of E

$$B_x = B\cos 60.0° = (40.0\text{ km})(0.500) = 20.0\text{ km}$$

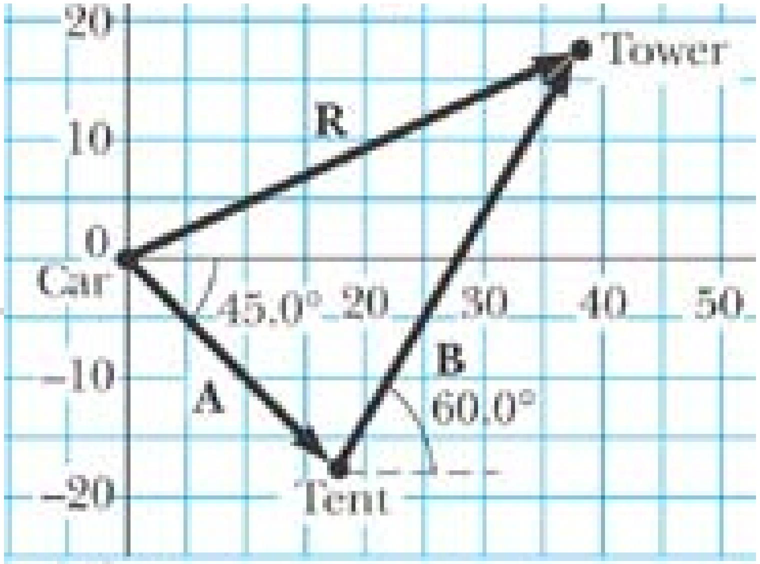
$$B_y = B\sin 60.0° = (40.0\text{ km})(0.866) = 34.6\text{ km}$$

(b) **R** = **A** + **B** has components :

$$R_x = A_x + B_x = 37.7\text{ km},\; R_y = A_y + B_y = 16.9\text{ km}$$

$$\mathbf{R} = (37.7\hat{\mathbf{i}} + 16.9\hat{\mathbf{j}})\text{ km} \quad \text{in unit - vector form}$$

Example 2, cont



- *Finalize:*
  - The units of **R** are km
    - ⇒ Reasonable for a displacement.
  - From the graphical representation, we estimate the final position to be at about (38 km, 17 km)
    - ⇒ Consistent with the components of **R** in our result
  - Both components of **R** are positive, putting the final position in the first quadrant of the coordinate system
    - ⇒ Consistent with the figure

Problem Solving Strategy

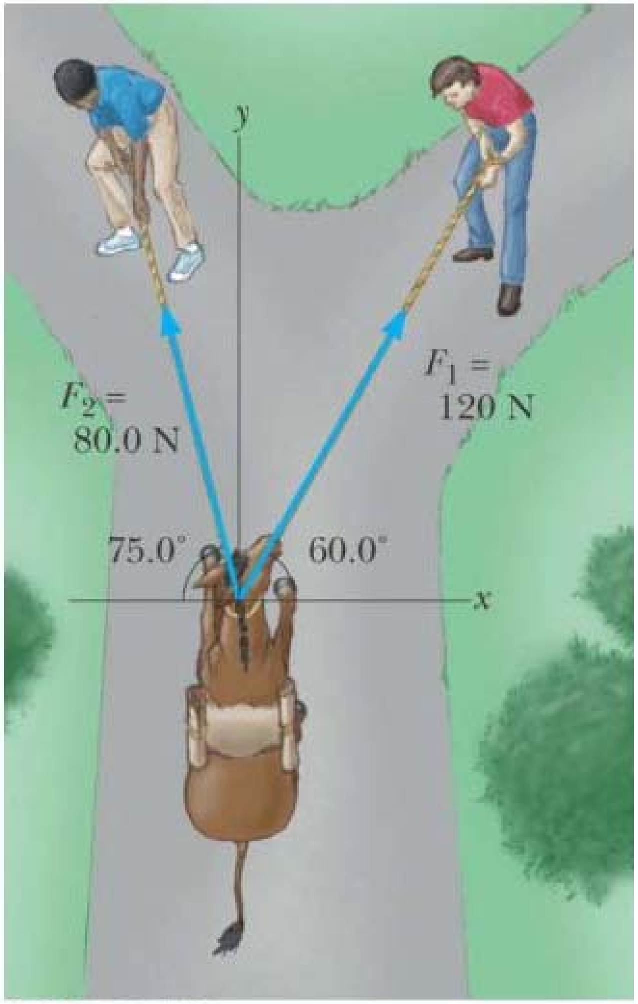
- Select a coordinate system
  - One that minimizes the number of components you need to deal with
- Draw a sketch of the vectors
  - Label each vector
- Draw the components of each vector
  - From them, find the components of the resultant vector
- Obtain the resultant vector
  - Use the Pythagorean theorem to find the magnitude and the tangent function to find the direction

Example 3

The helicopter view in the figure shows two people pulling on a stubborn mule.

Find (a) the single force that is equivalent to the two forces shown, and (b) the force that a third person would have to exert on the mule to make the resultant force equal to zero.

The forces are measured in units of Newtons (abbreviated N).



Example 3, cont

(a) $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$

$$\mathbf{F} = 120\cos(60.0°)\hat{\mathbf{i}} + 120\sin(60.0°)\hat{\mathbf{j}} - 80.0\cos(75.0°)\hat{\mathbf{i}} + 80.0\sin(75.0°)\hat{\mathbf{j}}$$

$$\mathbf{F} = 60.0\hat{\mathbf{i}} + 104\hat{\mathbf{j}} - 20.7\hat{\mathbf{i}} + 77.3\hat{\mathbf{j}} = (39.3\hat{\mathbf{i}} + 181\hat{\mathbf{j}})\,\text{N}$$

$$|\mathbf{F}| = \sqrt{39.3^2 + 181^2} = \boxed{185\ \text{N}}$$

$$\theta = \tan^{-1}\left(\frac{181}{39.3}\right) = \boxed{77.8°}$$

(b) $\mathbf{F}_3 = -\mathbf{F} = \boxed{(-39.3\hat{\mathbf{i}} - 181\hat{\mathbf{j}})\,\text{N}}$