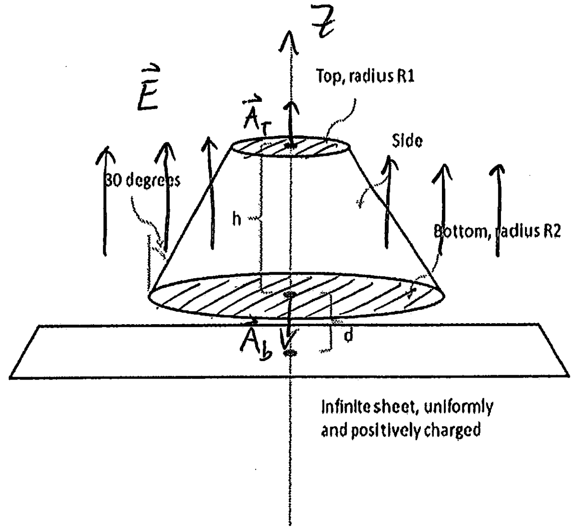
problem #1 [10 points]

A closed Gaussian surface is placed above an infinite sheet. The sheet is uniformly and positively charged with a charge per area of σ. The Gaussian surface has a base of radius R_2, slanted conical sides (angle is 30 degrees as depicted), and a top of radius R_1. The top and the bottom are parallel to the infinite sheet. The bottom surface is a height "d" above the sheet, and the entire object is a height "h".



Given d, h, R_1, R_2, and σ:

(a) [3 pts] What is the electric flux through the base?

$\Phi = \vec{E} \cdot \vec{A}$

$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{z}$ for infinite charged sheet ; $\vec{A}_{base} = -\pi R_2^2 \hat{z}$

$\Rightarrow \Phi_{base} = \frac{-\sigma \pi R_2^2}{2\epsilon_0}$

(b) [3 pts] What is the electric flux through the top?

$\vec{A}_{Top} = \pi R_1^2 \hat{z}$

$\Rightarrow \Phi_{top} = \frac{\sigma \pi R_1^2}{2\epsilon_0}$

(c) [4 pts] What is the electric flux through the side? Show your work and/or explain.

flux through curved surface has to be what flux came in the bottom but didn't come out the top b/c $\Phi_{ToT} = 0$ (no charge enclosed):

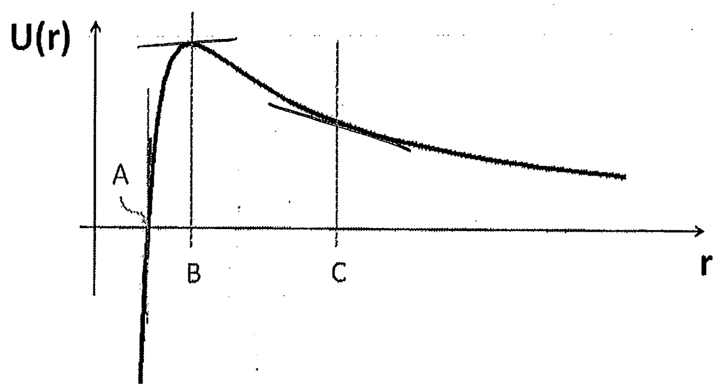
$\Phi_{ToT} = \Phi_{base} + \Phi_{top} + \Phi_{side} = 0$

$\Rightarrow \Phi_{side} = -\Phi_{base} - \Phi_{top}$

$\Phi_{side} = \frac{\sigma \pi}{2\epsilon_0}(R_2^2 - R_1^2)$

problem #2 [15 points]

The potential energy U of a positively charge particle with charge q_0 only depends on the radial distance away from some charge distribution. That is, all constant energy surfaces are spheres centered at r=0. The potential energy as a function of r is plotted here:



Given q_0>0 and U(r):

(a) [5 pts] At what radial distance is the magnitude of the force closest to zero (point A, B, or C)? Explain.

$\vec{F} = -\nabla U = -$ slope of $U(r)$

slope @ pt. B $\sim 0 \Rightarrow \vec{F}_B \sim 0$ is the smallest

(b) [5 pts] At what radial distance is the magnitude of the force the greatest (point A, B, or C)? Explain.

by same reasoning, slope @ pt. a is largest

$\Rightarrow \vec{F}_A$ is the largest

(c) [5 pts] What is the direction of the force when the particle is placed at a radial distance C? Explain.

$\vec{F} = -\frac{\partial U}{\partial r}\hat{r}$ (1 dimensional gradient)

by inspection, slope @ pt C is negative

$\Rightarrow \frac{\partial U}{\partial r} < 0$

$\Rightarrow \vec{F}_C > 0$

$\therefore$ force is in $+\hat{r}$ direction (consistent with visual intuition for a $+q_0$ particle, works just like gravity)

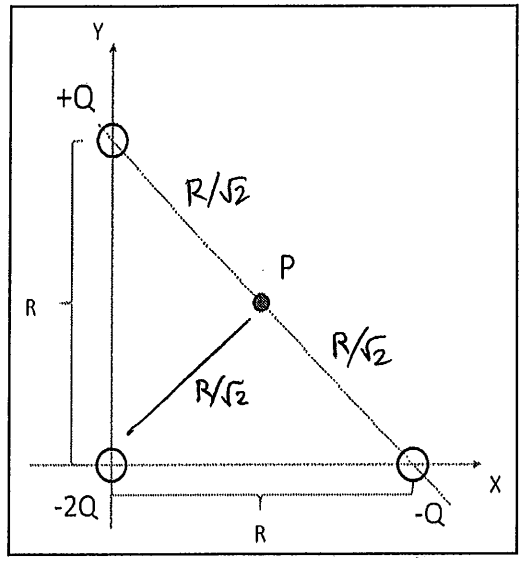
problem #3 [15 points]

45-45-90 Δ:

legs ratio is $1:1:\sqrt{2}$ for leg-leg-hyp

A negative charge of -2Q is placed at the origin, a negative charge of –Q is placed at x=R, and a positive charge +Q is placed at y=R. Point P is located half way between the +Q and –Q charge. [HINT: all distances from the 3 individual charges to point "P" are the same!]

Given the value of Q and R, and the configuration as depicted in the diagram:



all 3 charges are equidistant from P

(a) [4 pts] What is the electric potential at point P with respect to a point at infinity?

for a simple point charge distribution:

$$V = \Sigma \frac{kq}{r} \quad (\text{scalar sum})$$

$$\Rightarrow V_p = \frac{k}{(R/\sqrt{2})}(\cancel{Q} - \cancel{Q} - 2Q)$$

$$\Rightarrow V_p = -\frac{2\sqrt{2}\,kQ}{R}$$

(b) [4 pts] What is the external work required to bring a charge of +4Q from infinity to point P? Explain the significance of the sign.

$$W = \Delta U = q_0 \Delta V = q_0 (V_p - \cancel{V_\infty}^{\,0}) = 4Q\left(-\frac{2\sqrt{2}\,kQ}{R}\right) = \frac{-8\sqrt{2}\,kQ^2}{R}$$

(–) b/c work is done in direction opposing the motion as the particle moves in from ∞ toward P, to keep it from accelerating toward P.

(c) [4 pts] What is the electric field at point P (magnitude and direction)?

$$\vec{E}_{+q} = \frac{2kQ}{R^2}(\cos 45\,\hat{x} - \sin 45\,\hat{y})$$

$$\vec{E}_{-q} = \frac{2kQ}{R^2}(\cos 45\,\hat{x} - \sin 45\,\hat{y}) \quad (-Q \text{ changes sign overall})$$

$$\vec{E}_{-2q} = \frac{4kQ}{R^2}(-\cos 45\,\hat{x} - \sin 45\,\hat{y})$$

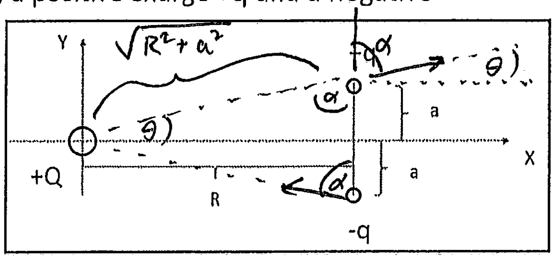
$$\Rightarrow E_x = 0 \;;\; E_y = -\frac{8kQ}{R^2}\left(\frac{\sqrt{2}}{2}\right) \Rightarrow \vec{E} = -\frac{4\sqrt{2}\,kQ}{R^2}\,\hat{y}$$

(d) [3 pts] What is the net force (magnitude and direction) on a +4Q charge placed at point P?

$$\vec{F} = q_0 \vec{E} = -16\sqrt{2}\,\frac{kQ^2}{R^2}\,\hat{y}$$

problem #4 [15 points + 5 point bonus question]

A charge of +Q is situated at the origin. The center of an electric dipole is located a distance "R" away on the x-axes. The electric dipole can be thought of as two charges, a positive charge +q and a negative charge −q, whose separation is always a constant distance, "2a".

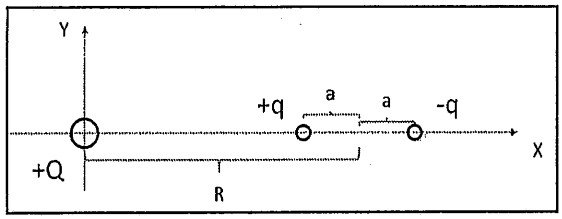


(a)

Given Q, q, R, and a, and for a dipole polarization vector that is perpendicular to the x-axes as depicted in the diagram to the right:

(a) [4 pts] What is the net force (magnitude and direction) on the dipole due to the charge at the origin? Explain and/or show work magnitudes are equal ; x-components cancel, but y-comps add ; each is $F_y = \frac{kqQ}{r^2}\sin\theta$; $r^2 = R^2+a^2$;

$\sin\theta = \frac{a}{\sqrt{R^2+a^2}} \Rightarrow \vec{F}_{net} = \frac{2kqQa}{(R^2+a^2)^{3/2}}\hat{y}$



(b)

Given Q, q, R, and a, and for a dipole polarization vector that is parallel to the x-axes as depicted in the diagram to the right:

(b) [4 pts] What is the net force (magnitude and direction) on the dipole due to the charge at the origin? Explain and/or show work.

mags : $F_{+q} = \frac{kqQ}{(R-a)^2}$; $F_{-q} = \frac{kqQ}{(R+a)^2}$; both are $\hat{x}$, but F_{-q} is in $-\hat{x}$ direction :

$\Rightarrow \vec{F}_{net} = kqQ\left(\frac{1}{(R-a)^2} - \frac{1}{(R+a)^2}\right)\hat{x}$

(c) [4 pts] What is the net torque (magnitude and direction) on the dipole? Explain and/or show work. $\vec{\tau} = \vec{a}\times\vec{F}$; $\vec{a} = \pm a\hat{x}$; $\vec{F}\sim\hat{x} \Rightarrow \vec{F} \parallel \vec{a}$

$\Rightarrow \vec{a}\times\vec{F} = 0 \Rightarrow \vec{\tau}_{net} = 0$

[3 pts] In which of the two above dipole configurations is the potential energy of the dipole the highest? Explain.

$U_a = kqQ\left(\frac{1}{\sqrt{R^2+a^2}} - \frac{1}{\sqrt{R^2+a^2}}\right) = 0$; $U_b = kqQ\left(\frac{1}{R-a} - \frac{1}{R+a}\right) > 0$

So U_b is higher, consistent with visual inspection... +q doesn't want to be closer to +Q   −q .....+q would be most stable w/ smallest U.

[BONUS + 5 pts] What is the net torque (magnitude and direction) on the dipole in the first (top most) figure? Show your work.

$\vec{\tau}_{+q} = \vec{a}\times\vec{F}_{+q} = aF_{\pm q}\sin\alpha = aF_{\pm q}\cos\theta$; $F_{\pm q} = \frac{kqQ}{R^2+a^2}$;

$\cos\theta = \frac{R}{\sqrt{R^2+a^2}} \Rightarrow \vec{\tau}_{+q} = \frac{kqQaR}{(R^2+a^2)^{3/2}}$ into page

$\vec{\tau}_{-q} = \vec{a}\times\vec{F}_{-q} = aF_{\pm q}\sin\alpha = \vec{\tau}_{+q} \Rightarrow \vec{\tau}_{net} = \frac{2kqQaR}{(R^2+a^2)^{3/2}}$ into page

(same direction also)

problem #5 [15 pts + 5 pt Bonus Question]

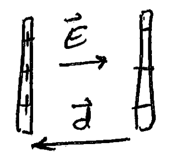
Two large circular parallel conducting plates of radii "R" with an air-filled separation of "d" are charged by a battery that applies a potential difference causing one of the plates to be charged to +Q and the other to –Q. Given d, R, and Q:

(a) [2 pt] What is the electric field between the plates (magnitude)?

between cap plates: $E = \frac{\sigma}{\epsilon_0}$; $\sigma = \frac{Q}{A} = \frac{Q}{\pi R^2}$ $\Rightarrow$ $E = \frac{Q}{\epsilon_0 \pi R^2}$

(b) [2 pt] What is the potential difference between the plates?

$\Delta V = -\vec{E}\cdot\vec{d} = \frac{Qd}{\epsilon_0 \pi R^2}$



$\vec{d}$ from lower to higher V

(c) [2 pt] What is the capacitance associated with the plates?

$C = \frac{Q}{\Delta V} = \frac{\epsilon_0 \pi R^2}{d}$

The plates are disconnected from the battery and completely immersed in a dielectric oil whose dielectric constant is K such that a charge +Q and –Q remain on the two conducting plates. Given d, R, Q, and K:

(d) [2 pt] What is the electric field between the plates (magnitude)?

$\epsilon_0 \rightarrow \epsilon = K\epsilon_0 \Rightarrow E = \frac{\sigma}{\epsilon} = \frac{Q}{K\epsilon_0 \pi R^2}$

(e) [2 pt] What is the potential difference between the plates?

$\Delta V = Ed = \frac{Qd}{K\epsilon_0 \pi R^2}$

(f) [2 pt] What is the capacitance associated with the plates?

$C = \frac{Q}{\Delta V} = \frac{K\epsilon_0 \pi R^2}{d}$

The plates while still completely immersed in the dielectric oil are then reconnected to the battery. Given d, R, Q, and K:

(g) [3 pts] What is the electric field between the plates (magnitude)? Explain and/or show work.

ΔV must be its original value (parts (a)-(c)); since its the same battery and separation d $\Rightarrow E = \frac{Q}{\epsilon_0 \pi R^2}$

[Bonus + 5 pts] Continuing under the conditions from part (g):

(h) [3 pts] What is the charge on the plates? Explain and/or show work.

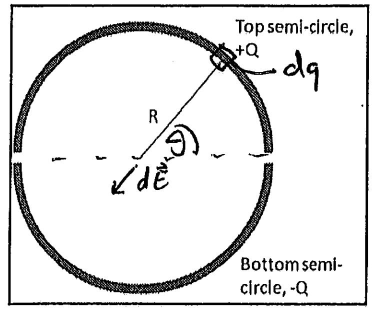
$C_0 \rightarrow C = KC_0$; $\Delta V = \Delta V_0 \Rightarrow Q = C\Delta V = KC_0 \Delta V_0 = KQ_0$

(i) [2 pts] How much energy is stored by the plates? Explain and/or show work.

$U = \frac{1}{2} C \Delta V^2 = \frac{1}{2}\left(\frac{K\cancel{\epsilon_0}\cancel{\pi R^2}}{\cancel{d}}\right)\left(\frac{Qd}{\epsilon_0 \pi R^2}\right)^2 = \frac{KQ^2 d}{2\epsilon_0 \pi R^2}$

problem #6 [15 points]

A thin insulating circular hoop is cut in half. The top half is charged with a positive charge of +Q and the bottom half is charged with negative charge of –Q. Both the top and bottom semicircles have a uniform charge per length. The two halves are then placed very close to one another to form a circular hoop. Given Q and R:



(a) [8 pts] What is the electric field at the center of the hoop (magnitude and direction)? Show your work.

x-components will cancel due to symmetry, y-compo will add:

$E_y = E \sin\theta$ (for top and bottom) $= \frac{k}{R^2}\int dq \sin\theta$;

$dq = \lambda ds = \lambda R \, d\theta$; $\lambda = \frac{Q}{\pi R} \Rightarrow \lambda R \, d\theta = \frac{Q}{\pi} d\theta$

$$\Rightarrow E_y = \frac{kQ}{\pi R^2}\int_0^{\pi} \sin\theta \, d\theta = \frac{kQ}{\pi R^2}\left(-\cos\theta \Big|_0^{\pi}\right) = \frac{kQ}{\pi R^2}(1+1)$$

$2E_y$ downward is total field:

$$\Rightarrow \vec{E} = -\frac{4kQ}{\pi R^2}\hat{y}$$

(b) [7 pts] What is the potential at the center of the hoop (compared to the potential at infinity)? Show your work.

Since R is constant, integral of $V = \frac{k}{R}\int dq$ is trivial:

$\Downarrow$

$V_{top} = \frac{kQ}{R}$

$V_{bottom} = -\frac{kQ}{R}$

$\Rightarrow V_{tot} = 0$

$dq = \lambda R \, d\theta = \frac{Q}{\pi} d\theta$

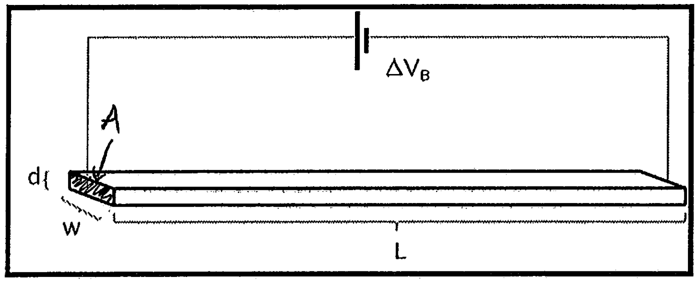
$V = \frac{kQ}{\pi R}\int_0^{\pi} d\theta = \frac{kQ}{\pi R}(\pi - 0)$

$V = \frac{kQ}{R}$

Same for top and bottom, except $-Q$

problem 7 [15 points]

A long ohmic conducting bar of width w, thickness d, and length L is connected to a battery that applies a potential difference of ΔV_B across the entire length of the bar. The conducting bar has a uniform resistivity of ρ. Assume a uniform current density throughout the entire conducting bar.



Given ΔV_B, ρ, d, w, and L:

(a) [2 pts] What is the total current supplied by the battery?

$R = \frac{\rho L}{A}$; A is cross-sectional area that the current sees $\Rightarrow R = \frac{\rho L}{wd}$; $I = \frac{\Delta V_B}{R}$

$I = \frac{\Delta V_B wd}{\rho L}$

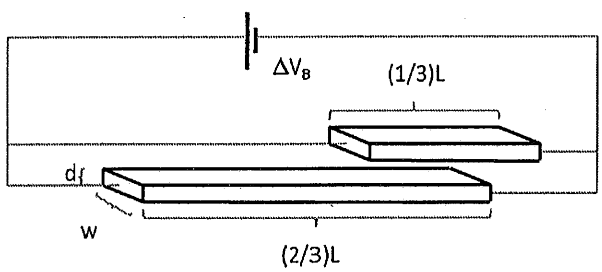
(b) [2 pts] What is the current density in the conducting bar?

$$J = \frac{I}{A} = \frac{\Delta V_B \cancel{wd}}{\rho L \cancel{(wd)}} = \frac{\Delta V_B}{\rho L}$$

(c) [2 pts] How much heat is generated in the conducting bar during a time "t"?

$$E = P\cdot t = V_B I t = \left(\frac{\Delta V_B wd}{\rho L}\right)\Delta V_B t = \frac{\Delta V_B^2 wdt}{\rho L}$$

The conducting bar is now cut into two pieces, one of length (1/3) L and the other of length (2/3) L, and wired to the battery as shown in the diagram to the right. Note that the resistance in the two segments is (1/3) R and (2/3) R, respectively, where R is the total resistance of the uncut bar.



Given ΔV_B and R:

(a) [3 pts] What is the total current supplied by the battery?

$$\frac{1}{R_{eq}} = \frac{1}{\frac{1}{3}R} + \frac{1}{\frac{2}{3}R} \Rightarrow \frac{(\frac{1}{3}R)(\frac{2}{3}R)}{\frac{1}{3}R + \frac{2}{3}R} = R_{eq} = \frac{\frac{2}{9}R^{\cancel{2}}}{\cancel{R}} = \frac{2}{9}R$$

$$I_T = \frac{\Delta V_B}{R_{eq}} = \frac{9}{2}\frac{\Delta V_B}{R} = \frac{9}{2}\frac{\Delta V_B wd}{\rho L}$$

(b) [3 pts] What is the current through the shortest conducting bar?

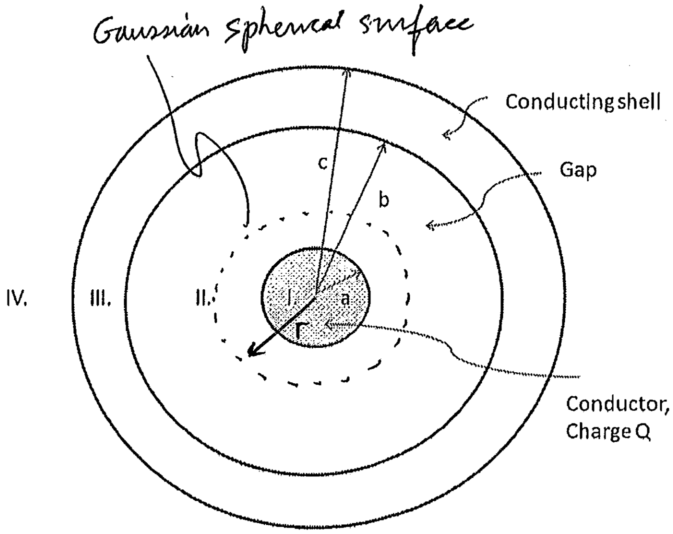
ΔV_B is potential across each parallel branch ; ie each segment of bar

$$I_{1/3} = \frac{\Delta V_B}{\frac{1}{3}R} = \frac{3\Delta V_B}{R} = \frac{3\Delta V_B wd}{\rho L}$$

(c) [3 pts] What is the total power dissipated by both conducting bars?

$$P = I_T \Delta V_B = \frac{9}{2}\frac{\Delta V_B^2 dw}{\rho L}$$

problem 8 [15 points + 10 point bonus question]



A solid conducting sphere (region I.) of radius "a" with a positive net charge of +Q is surrounded by a concentric conducting shell (region III.) of inner radius "b" and outer radius "c" which is neutral (no net charge). A gap (region II.) exists between the two conductors. Region IV is filled with air.

Given Q, a, b, and c, and a gap filled with air,

(a) [7 pts] Find the electric field in region I., II., III., and IV.. Show your work and/or explain.

E inside conductor is zero, so:

$E_I = E_{III} = 0$

use Gauss's law for others:

$\oint \vec{E}_{II} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E_{II}(4\pi r^2) = \frac{Q}{\epsilon_0} \Rightarrow \vec{E}_{II} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} = \frac{kQ}{r^2}\hat{r}$

for IV : same total charge is enclosed, so field is the same :

$\Rightarrow \vec{E}_{IV} = \frac{kQ}{r^2}\hat{r}$

(b) [8 pts] What is the induced charge per area on the conducting shell's inner surface (at radius b)? Explain and/or show work.

using a Gaussian surface inside region III, we know $\vec{E} = 0 \Rightarrow Q_{enc} = 0$

$\Rightarrow$ Q on inner sphere + Q_b induced $= 0$

$\Rightarrow Q_b = -Q$

$\sigma_b = \frac{Q_b}{A_b} = \frac{-Q}{4\pi b^2}$

BONUS [10 pts]: Consider that the gap (all of region II.) is filled with a dielectric that has a dielectric constant of K. What is the induced charge per area on the dielectric's inner surface (at radius a) as well as the dielectric's outer radius (at radius b)? Derive your results! Show your work.

$\epsilon_0 \to \epsilon = K\epsilon_0 \Rightarrow E_{II} = \frac{Q}{4\pi\epsilon_0 K r^2}$; $\oint E_{II}\, dA = \frac{Q_{TOT}}{\epsilon_0} = \frac{Q_{a\ induced}}{\epsilon_0} + \frac{Q}{\epsilon_0}$

$\Rightarrow \frac{Q}{\cancel{4\pi\epsilon_0}Kr^2}(\cancel{4\pi r^2}) = \frac{Q_a + Q}{\cancel{\epsilon_0}} \Rightarrow Q_a = Q\left(\frac{1}{K} - 1\right)$ or $Q_a = -Q\left(\frac{K-1}{K}\right)$

by the same reasoning as part (b), $Q_b = -Q_a$

$\Rightarrow \sigma_a = -\frac{Q}{4\pi a^2}\left(\frac{K-1}{K}\right)$; $\sigma_b = \frac{Q}{4\pi b^2}\left(\frac{K-1}{K}\right)$