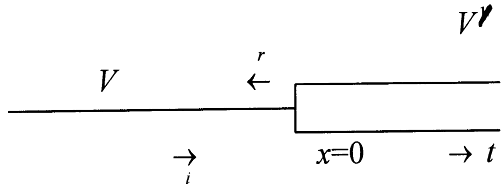
STANDING WAVES/STRING INSTRUMENTS

PLUS PHASE CHANGE DURING REFLECTION



We have learned that if two strings meet at x=0, then an incident wave

$$Y_i = A_i Sin(\kappa x - \omega t)$$

Where $\frac{w}{k} = v$

Will, on arriving at x=0, give rise to two waves

Reflected $Y_r = A_r Sin(\kappa x + \omega t)$ and

Transmitted $Y_t = A_t Sin(\kappa' \ x - \omega t)$

Where $\frac{w}{k'} = v'$

Note: FREQUENCY DOES NOT CHANGE

Further,

$$\frac{A_r}{A_i} = \frac{v - v'}{v + v'}$$

$$\frac{A_t}{A_r} = \frac{2v'}{v + v'}$$

Interesting situation arises if $v' \to 0$, that is, string on the right is like a 'wall" or equivalently the end of the string on left is "fixed" at x=0. In that case

$$\frac{A_r}{A_i} = 1 \ ; \ \text{So,} \ \begin{array}{l} Y_i = A_i Sin(\kappa x - \omega t) \\ Y_r = A_i Sin(\kappa x + \omega t) \end{array}$$

Now we have two waves on the string at the same time and to handle it, we use the principle of SUPERPOSITION. Since a wave is just a disturbance or a deviation, it is perfectly legitimate to have many simultaneous disturbances at the same point in space. The net effect is that one must algebraically add all of the disturbances $D = \sum D_i$, where

$$D_i = A_i Sin(\kappa_i x \mp \omega_i t)$$

and $\frac{\omega_i}{\kappa_i} = v$

So, that total wave will be

$$Y = Y_i + Y_r$$

$$= A_i \ Sin(\kappa x - \omega t) + A_i \ Sin(\kappa x - \omega t)$$

using the trigonometric identity

$$Sin(\theta_1 \pm \theta_2) = Sin\theta_1 \ Cos\theta_2 \pm Cos\theta_1 \ Sin\theta_2$$

we get

$$y = 2 A_i \ Sin\kappa x \ Cos\omega t$$

$$= 2 A_i \ Sin\frac{2\pi x}{\lambda} Cos\,\omega t$$

and you see that y=0 if

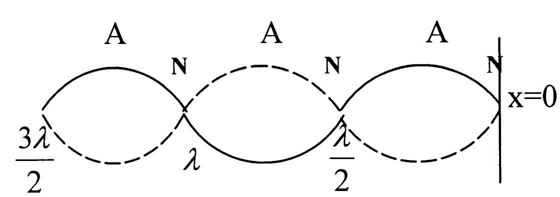
$$x = 0, \frac{-\lambda}{2}, -\lambda, \frac{-3\lambda}{2}, etc.$$

That is, there is NO MOTTION AT ALL AT SOME POINTS OF THE STRING. These points are called NODES.

In between two nodes, that is, at

$$x = \frac{-\lambda}{4}, \frac{-3\lambda}{4}, \frac{-5\lambda}{4}, etc.$$

The string vibrates with twice the amplitude. These points are termed ANTINODES.

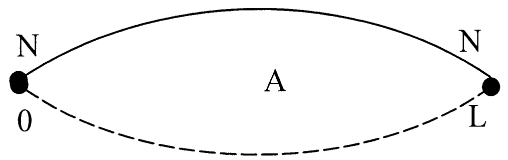


This is how the string will look where the i and r waves ware both present.

The case of most interest arises when the wire is fixed at both ends (as in musical instruments).

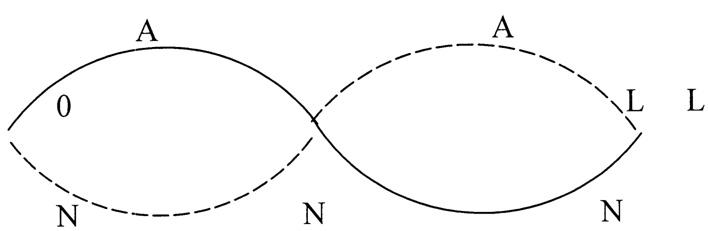
Because of what we learned above, there must be a node at either end and there must be a node every $\frac{\lambda}{2}$ as well. This requires that the wire can vibrate in only certain specific MODES such as:

FIRST HARMONIC, *n*=1



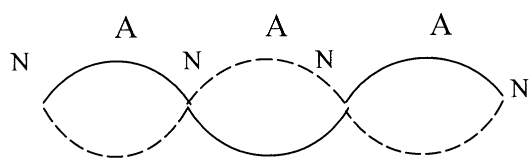
$$\frac{\lambda_1}{2} = L$$

SECOND HARMONIC, *n*=2



$$2\left(\frac{\lambda_2}{2}\right) = L$$

THIRD HARMONIC, *n*=3



$$3\left(\frac{\lambda_3}{2}\right) = L$$

That is, the wavelengths λ_n of the modes must obey

$$\frac{n\lambda_n}{2} = L$$

or

$$\lambda_n = \frac{2L}{n}$$

n=1,2,3, etc. or in words, only those modes can occur in which there is an integer number of "half wavelengths" fitting on the wire.

The modes with $n \geq 2$ are called *Harmonics* of the fundamental mode. That word comes from musical ethos.

Next,

$$v = \sqrt{\frac{F}{\mu}}$$

So the frequencies of these modes will

$$f_n = \frac{v}{\lambda_n}$$

$$= \frac{n}{2L}\sqrt{\frac{F}{\mu}}$$

And this Equation describes all string instruments. To be precise:

1) When you tighten a string, the note goes "up" because v increases for a given λ (length).
2) The shorter the string the higher the note.
3) If you look inside a piano you will notice that the lowest notes have very thick strings. Here, a high μ is used to reduce v and thereby lower f.
4) If you are "playing" a single string on the sitar or guitar you must move close to the lower end to get a higher note as this reduces the length of the string where you are plucking.

5) If you pull the string sideways you can get subtle variations in the frequency. This is most often used by sitar players. It works because you can vary the tension by small amounts. Such subtle variations are also accomplished by imaginative bowing of the violin/viola/bass/fiddle.

APPENDIX

PHASE CHANGES ON REFLECTION

When

$$v' = 0$$
$$Y_i = A_i \, Sin(\kappa x - \omega t) \quad \frac{A_r}{A_i} = 1$$
$$Y_r = A_i \, Sin(\kappa x + \omega t)$$

Note that reflected wave is "born" when incident wave arrives at x=0. We can compare the phases at

$$x = 0$$
$$Y_i = A_i \, Sin(-\omega t)$$
$$= -\, A_i \, Sin \; \omega t$$
$$= +\, A_i \, Sin(\omega t + \pi)$$
$$Y_r = A_i \, Sin \; \omega t$$

So you see that during reflection at a fixed end there is a phase change of π. If a "crest" arrives, it leaves as a "trough" and vice versa.

The other extreme case

If $v' >> v$

$$\frac{A_r}{A_i} \cong -1, \frac{A_t}{A_i} = 2 \, .$$

Since energy transport is $\eta = \frac{1}{2} A^2 \omega^2 \frac{F}{v}$ and $v' >> v$, η_T is very small. That is very little energy is transmitted into the wire on the right.

For wire on the left at x=0, $A_r = -A_i$

hence

$$Y_i = -\, A_i Sin\omega t$$

also

$$Y_r = -\, A_i Sin\omega t$$

So, no change of phase in this case.

When a crest arrives it leaves as a crest.

Summary

Reflection at a "fixed" end ⇒ phase change of π

Reflection at an "open" end ⇒ No phase change.

(we return to this in more detail later)