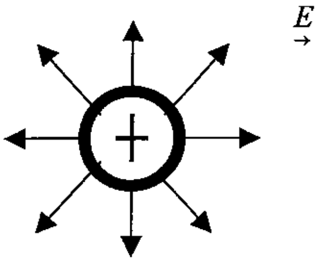
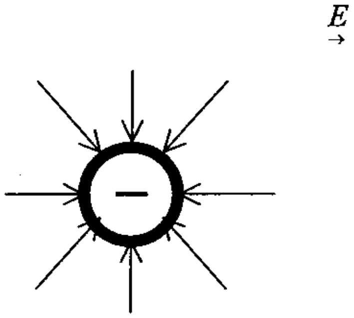
FLUX OF $\vec{E}$– FIELD: GAUSS' LAW

We have used the analogy of water flowing into (out of) a sink (source) to imagine the $\vec{E}$– *field* surrounding a point charge.





Just as we talk of the amount of water flowing through an area, we now talk of the FLUX of $\vec{E}$ as the quantity

- $\Delta\Phi_E = \vec{E}\bullet\Delta\vec{A} = E\Delta A\ Cos(\vec{E},\hat{n})$

as a measure of the "amount" of $\vec{E}$– *field* "flowing" out of or into a surface of area $\Delta\vec{A} = \Delta A\hat{n}$

Example

Take area ΔA lying in xy-plane

$\Delta\vec{A} = \Delta A\hat{n}$

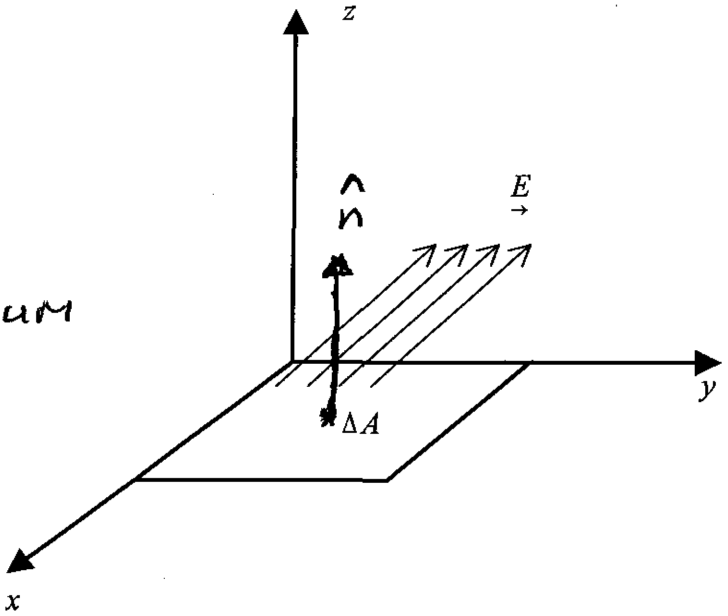
$= \Delta A\hat{z}$

Take $\vec{E} = E_y\hat{y} + E_z\hat{z}$

Notice:

FLUX IS MAXIMUM when $\vec{E} \parallel \hat{n}$

FLUX IS ZERO WHEN $\vec{E} \perp \hat{n}$



Thus

$$\Delta\Phi_E = E\,\Delta A\,Cos(\underset{\rightarrow}{E},\hat{z})$$

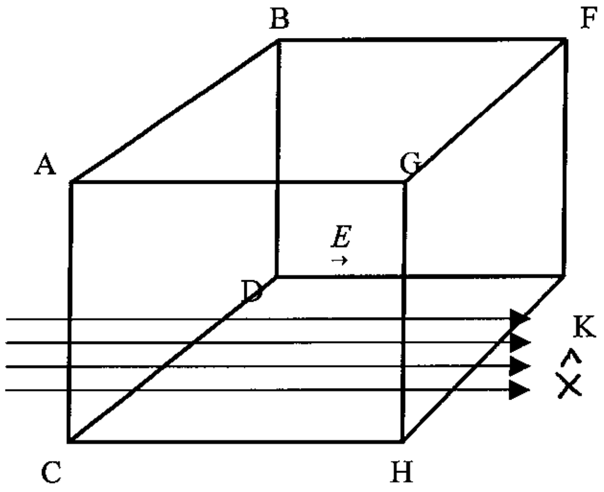
or

$$\Delta\Phi_E = (\Delta A)E_z$$

That is only component of $\underset{\rightarrow}{E}$ parallel to $\hat{z}$ contributes to flux of $\underset{\rightarrow}{E}$ through ΔA. [To go through a door you must travel along its normal]

<u>Example</u>:

Consider a cube. Let $\underset{\rightarrow}{E} = E\hat{x}$



NO SOURCE OR SINK INSIDE CUBE

Flux of $\underset{\rightarrow}{E}$ is non-zero only over faces ABCD and FGHK.
Over ABCD flux is into Cube
Over FGHK flux is out of Cube

TOTAL FLUX THROUGH CUBE=0 AS IT MUST BE BECAUSE THERE IS NO SOURCE OR SINK INSIDE IT. EVERY LINE THAT COMES IN MUST LEAVE [*LINES STOP (START) AT SINKS (SOURCES) ONLY*].

Imp

So it is not surprising that Gauss' Law says: THE TOTAL FLUX OF THE $\underset{\rightarrow}{E}$-FIELD <u>THROUGH</u> A <u>CLOSED</u> SURFACE IS DETERMINED SOLELY BY THE SOURCES (+CHARGES) AND SINKS (-CHARGES) IN THE VOLUME ENCLOSED BY THE SURFACE.

Mathematically, $\boxed{\Sigma_c \underset{\rightarrow}{E}\bullet\underset{\rightarrow}{\Delta A} = \Sigma_c E\perp\Delta A = \frac{1}{\varepsilon_0}\Sigma Q_i}$ (I)

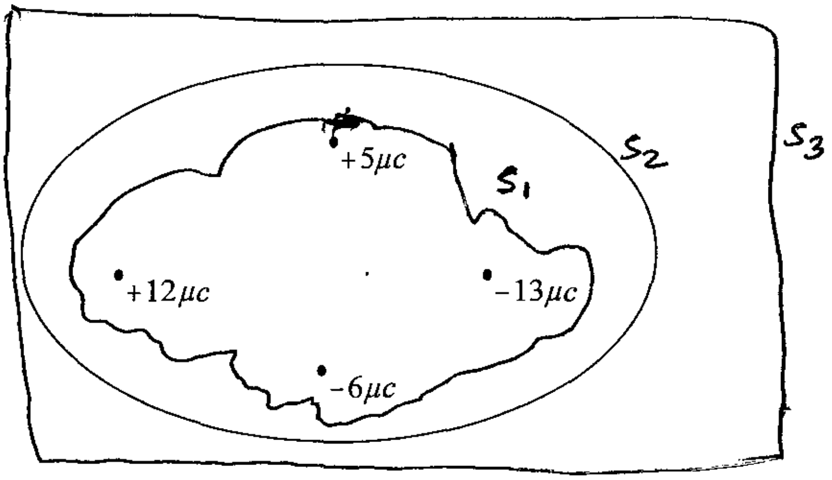
Where $\frac{1}{4\pi\varepsilon_0} = k_e = 9x10^9\frac{N-m^2}{c^2}$, $\varepsilon_0 = 9x10^{-12}\,F/m$

The Σ_c on the left hand side is sum of the flux over <u>all</u> parts of the closed surface. ΣQ_i is the algebraic sum of all the enclosed charges.

<u>Example:</u>

In general, Gauss' law would be very difficult to use to calculate $\underset{\rightarrow}{E}$. All it can give us is the total flux once the Q_i's are known. Here are 4 charges. If you draw any surface enclosing all of them we can write down the total flux of $\underset{\rightarrow}{E}$ immediately.

S_1, S_2, S_3 are any three closed surfaces drawn around our four charges. In Each Case



$$\Sigma_c \underset{\rightarrow}{E}\bullet\underset{\rightarrow}{\Delta A} = \frac{1}{9x10^{-12}}\left[5x10^{-6} + 12x10^{-6} - 6x10^{-6} - 13x10^{-6}\right]$$
$$= -\,2.2x10^5\,\frac{N-m^2}{C}$$

but it tells you nothing about the $\underset{\rightarrow}{E}$-field at any point on any Surface.

However, there are some special cases where we can use this law to calculate $\underset{\rightarrow}{E}$-field in one step. *[Please look at notes from Phys 121 to recall similar results for Gravitational field.]*

The calculation depends crucially on recognizing the symmetry of the problem and choosing the Gaussian Surface in such a way that the sum on the left of Eq. I can be calculated right away.

<u>Example 1</u>:

Point charge $+Q$ located at the origin, r=0. Because of spherical symmetry about r=0, the $\underset{\rightarrow}{E}$ -field cannot depend on angle. It must be radial (*along* $\hat{r}$) and be a function of $\underset{\rightarrow}{r}$ only. We should choose for our surface a sphere of radius r centered at r=0. Then I) E has the same magnitude at all points on the surface of the sphere and II)

$$\underset{\rightarrow}{E} \bullet \Delta \underset{\rightarrow}{A} = E(r)4\pi r^2 = \frac{Q}{\varepsilon_0}$$

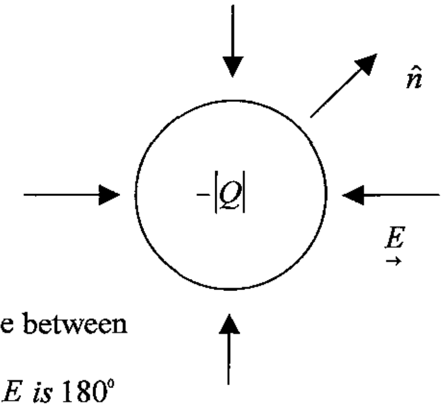
because

$$\underset{\rightarrow}{E} \| \hat{r}$$

$$\underset{\rightarrow}{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{k_e Q}{r^2}\hat{r}$$

Example 2:

$-|Q|$ Located at r=0. Again $\underset{\rightarrow}{E}$ can only be a function of r and directed along $-\hat{r}$ so (*FLUX is inward*) $-E \bullet 4\pi r^2 = \dfrac{-|Q|}{4\pi\varepsilon_0}$

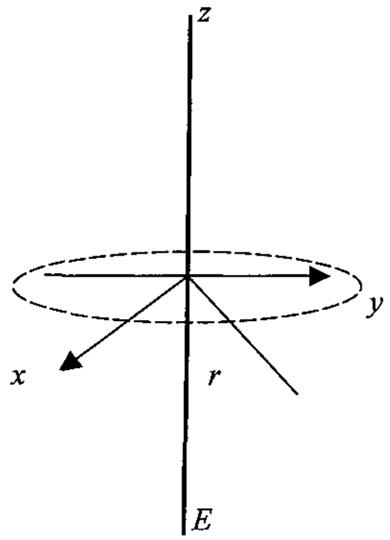


$$\underset{\rightarrow}{E}(r) = \frac{-|Q|}{4\pi\varepsilon_0}\hat{r}$$

angle between $\hat{n}$ & $\underset{\rightarrow}{E}$ *is* 180°

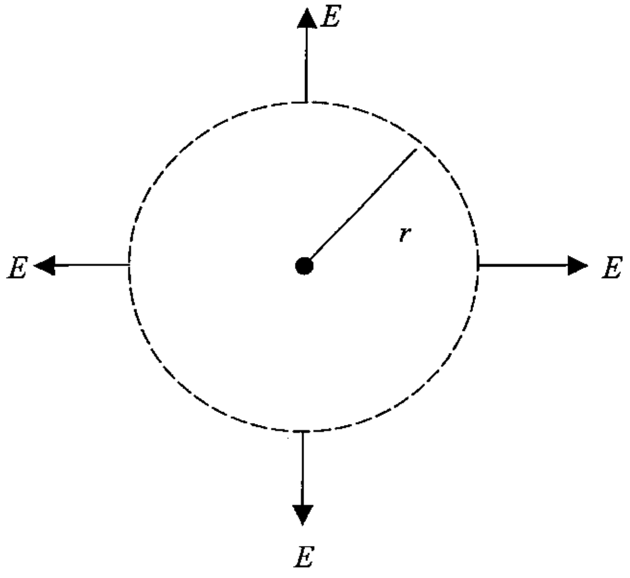
Example 3:

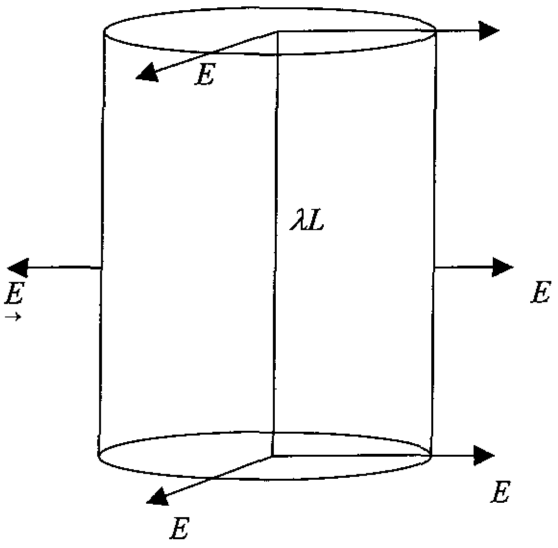
Long Line of charge: λ C/m along $\hat{z}$. Now there is cylindrical symmetry about z-axis.
Field cannot depend on z.
Field cannot depend on Angle.
Must be a fn of r only and directed along r.



λ C/m of charge on line

Choose as surface cylinder of length L, radius r, axis of cylinder on z-axis. Then $\sum_c \underset{\rightarrow}{E}\bullet\Delta\underset{\rightarrow}{A} = E(r)2\pi rL$. No contribution from End surfaces as $\underset{\rightarrow}{E}$ is parallel to them.





Also

$$\sum Q_i = \lambda L$$

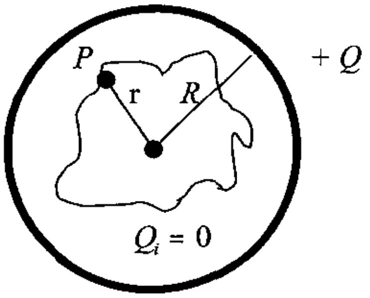
so

$$E(r)\cdot 2\pi rL = \lambda L$$

$$\underset{\rightarrow}{E}(r) = \frac{\lambda}{2\pi r\varepsilon_0}\hat{r}$$

Example 4:

Hollow sphere (spherical shell) has charge $+Q$ on its surface, radius R and is centered at r=0. Again symmetry about r=0, requires that $\underset{\rightarrow}{E}\|\hat{r}$ and is a function r only.



First, consider r<R. Choose any surface through r. $r < R$ $\sum_c \underset{\rightarrow}{E}\cdot\Delta A = 0$ → No enclosed charge.

If $\sum_c \underset{\rightarrow}{E}\cdot\Delta\underset{\rightarrow}{A} = 0$ for any and every surface as long as r<R, it implies $\underset{\rightarrow}{E} = 0$

$$\sum_{c} \underset{\rightarrow}{E} \cdot \underset{\rightarrow}{\Delta A} = E(r) 4\pi r^2$$

If $r>R$; the appropriate surface is a sphere of radius r and now [illegible]

Hence,

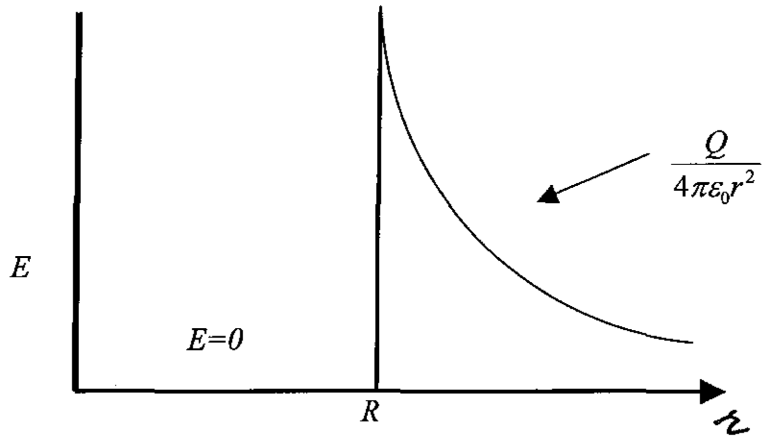
$$\underset{\rightarrow}{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$= \frac{k_e Q}{r^2}\hat{r}$$

$r > R$

as if the shell were replaced by a single charge Q located at its center ($r=0$). That is, r must be measured from the center of the shell.

Sph. Shell: Q on Shell



Note: As one goes from $r<R$ to $r>R$, the $\underset{\rightarrow}{E}$-field jumps by

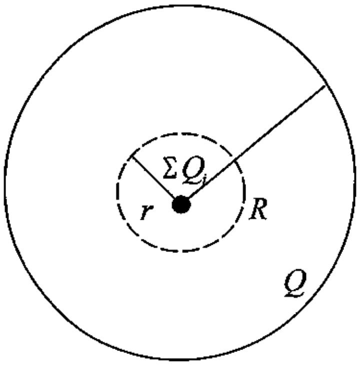
$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\sigma}{\varepsilon_0} \qquad \left[\sigma = \frac{Q}{4\pi r^2}\right]$$

where σ = charge density on the surface of the shell. Crossing a sheet of charge $\underset{\rightarrow}{E}$ jumps!

Ex5: Insulating Solid Sphere

Charge Q distributed uniformly over a sphere of radius R which is located with its center at $r=0$. First define charge density $\rho = \frac{Q}{\frac{4\pi}{3}R^3}$ $[\rho = \text{Rho}]$

Again, spherical symmetry obtains about $r=0$.



Now, if $r<R$.

$$\sum_{c} \underset{\rightarrow}{E} \bullet \Delta \underset{\rightarrow}{A} = E(r)4\pi r^2$$

$$\Sigma Q_i = \rho \cdot \frac{4\pi}{3} r^3$$

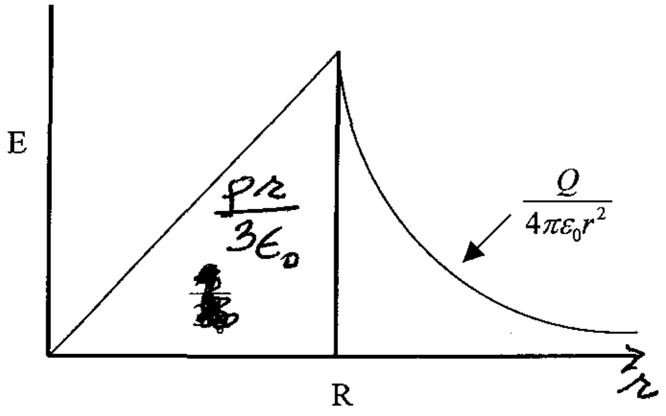
so for $r < R$.

$$E(r)4\pi r^2 = \frac{\rho}{\varepsilon_0} \cdot \frac{4\pi}{3} r^3$$

$$\underset{\rightarrow}{E} = \frac{\rho}{3\varepsilon_0} r\,\hat{r} \qquad r < R$$

If $r>R$ all of Q contributes so

$$\underset{\rightarrow}{E} = \frac{Q}{4\pi\,\varepsilon_0 r^2}\hat{r} \qquad r > R$$



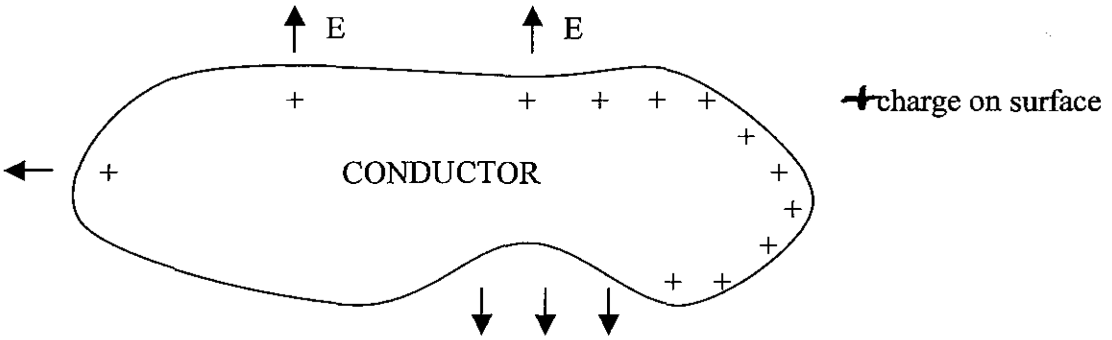
INSULATING SPHERE UNIFORMLY CHARGED with $\rho = \frac{Q}{\frac{4\pi}{3} R^3}$

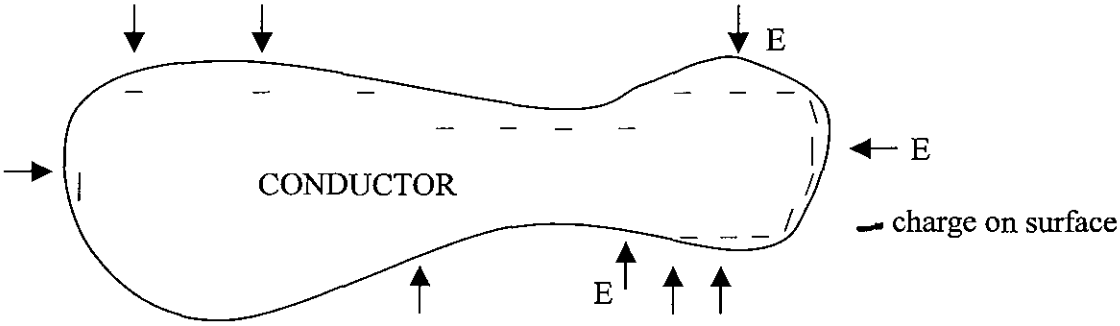
<u>Ex #6</u>

Conductor under stationary conditions: charge <u>NOT</u> allowed to move. If charge has to be immobile the field inside must be <u>zero</u> at every point. This is possible only if <u>Q=0</u> at every point inside the conductor. So under stationary conditions charge can reside <u>ONLY ON</u> the surface of the conductor <u>consequences</u>:

Q on conducting sphere ≡ Hollow spherical charge

Further, $\vec{E}$ at surface must be perpendicular to surface otherwise charges will start moving along surface.

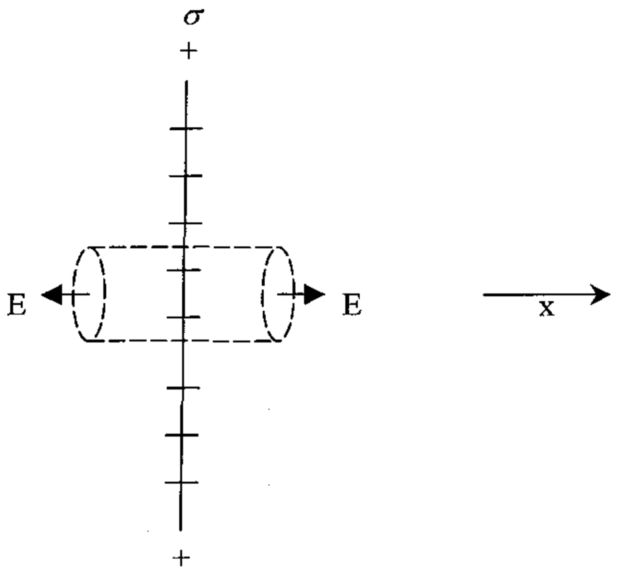




<u>Notice</u> In both cases force on charge is outward, that is, charge is bound to the surface.

Ex 7:

Sheet of charge ⊥ x-axis carries $+\sigma\ C/m^2$ of charge. Sheet located at x=0. Look at it end-on Recall that we have shown if two equal charges are at $+y$ and $-y$ the $\underset{\to}{E}$ field is purely along $\hat{x}$.



So here $\underset{\to}{E}$ along $+\hat{x}$ on right $-\hat{x}$ on left. Choose cylinder as Gaussian Surface.

$$\Sigma_c \underset{\to}{E}\bullet\Delta\underset{\to}{A} = E\pi r^2 + E\cdot\pi r^2$$

$$= E\cdot 2\pi r^2$$

and

$\Sigma Q_i = \sigma\pi r^2$ [charge enclosed by cylinder]

so

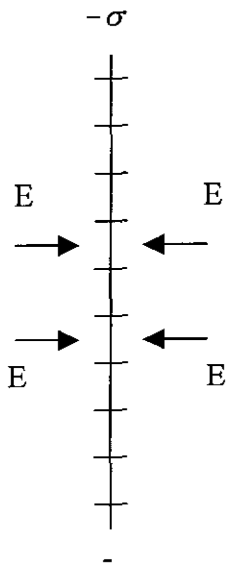
$$2E\pi r^2 = \sigma\cdot\pi r^2$$

$$\underset{\to}{E} = +\frac{\sigma}{2\varepsilon_0}\hat{x} \qquad x > 0$$

$$= -\frac{\sigma}{2\varepsilon_0}\hat{x} \qquad x < 0$$

Ex 7': Sheet carries $-|\sigma| C/m^2$

Then
$$\underset{\rightarrow}{E} = -\frac{\sigma}{2\varepsilon_0}\hat{x} \quad x > 0$$
$$= +\frac{\sigma}{2\varepsilon_0}\hat{x} \quad x < 0$$



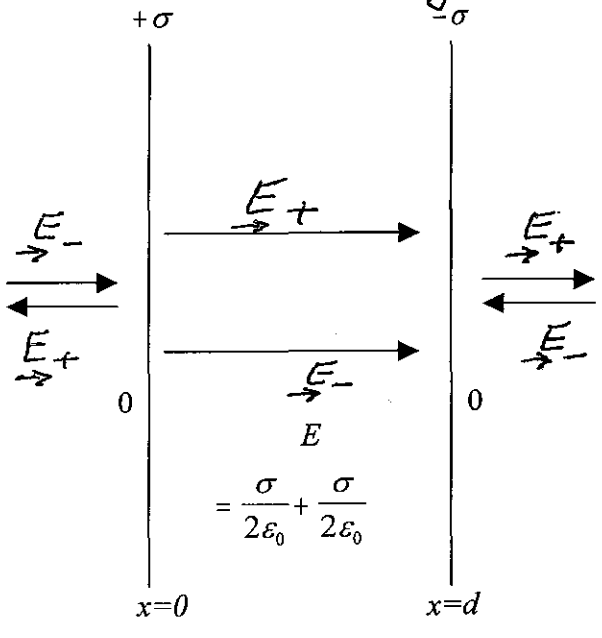
Ex 8: Sheet at $x=0$, $\sigma\, C/m^2$
Sheet at $x=d$, $-\sigma\, C/m^2$

Now the $\underset{\rightarrow}{E}_+$ and $\underset{\rightarrow}{E}_-$ fields will add vectorially. Hence

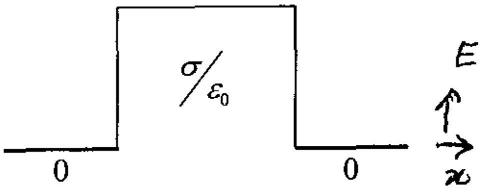
$\underset{\rightarrow}{E} = 0, x < 0$

$\underset{\rightarrow}{E} = \frac{\sigma}{\varepsilon_0} \quad 0 < x < d$

$\underset{\rightarrow}{E} = 0 \quad x > d$



Net $\underset{\rightarrow}{E}$ field:



Again $\underset{\rightarrow}{E}$ jumps on crossing sheet of charge