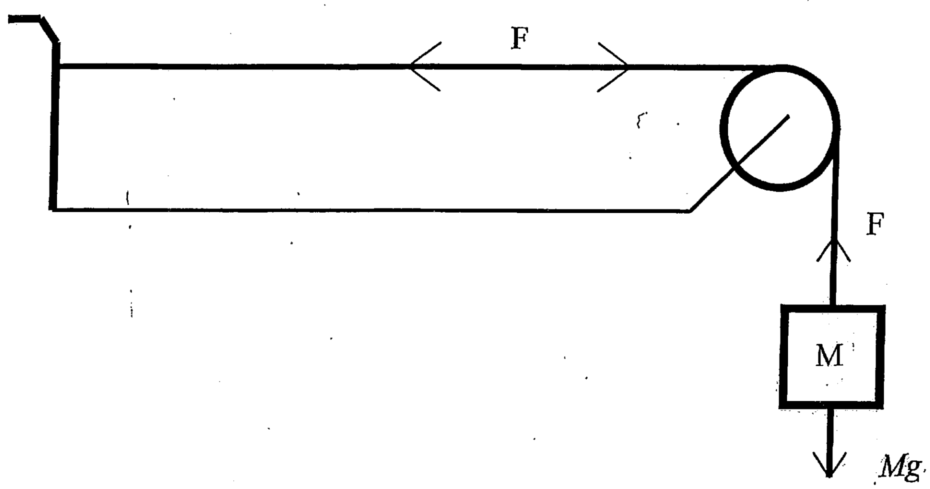
SPEED OF TRANSVERSE PULSE ON A STRETCHED STRING, PERIODIC WAVE, ENERGY TRANSPORT

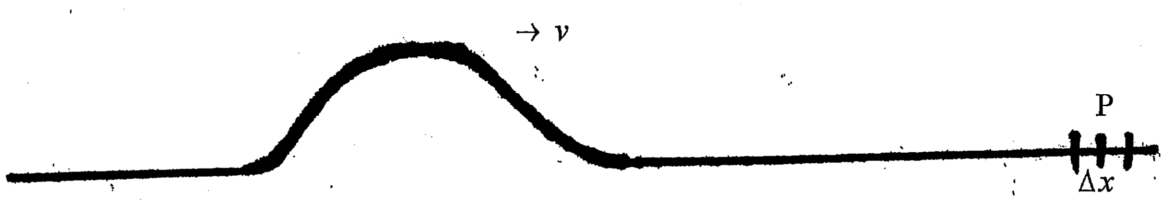
We now know that we can describe a transverse periodic wave of wavelength λ and a frequency f by the equation $y = A\,Sin(kx - \omega t)$ with $k = 2\pi/\lambda$ and $\omega = 2\pi f$ while $\vec{A} \perp \hat{x}$ with $\omega = vk[\text{same as } v = \lambda f]$

To generate a "pulse" we need to sum up many, many periodic waves with different wavelengths, frequencies and amplitudes. Experimentally, all we need is to take a string of length L and mass m tie its one end, pass the other over a pulley and hang a mass M.

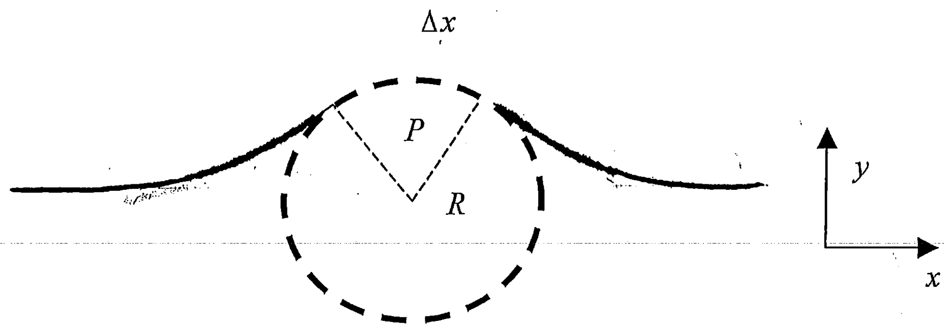


We define linear density $\mu = \dfrac{m}{L}$

The string will develop tension $F=(Mg)$ everywhere. We will make the string very long, so we do not need to worry about what happens at the ends as of yet. If we "tweak" it, we can observe a pulse such as shown below traveling along it.

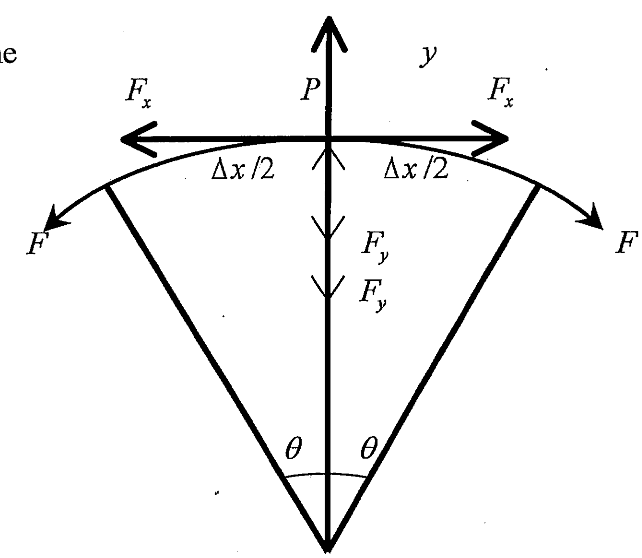


We will keep the amplitude small. Let us concentrate on a small piece of length Δx and ask what happens when the pulse comes along. As is clear Δx is lying there minding its own business when the pulse arrives and Δx must travel on a curved path to participate in the passage of the pulse. Indeed we can imagine that Δx is carried around a circle of radius R at speed υ.



Since Δx has a mass of $\mu\Delta x$ it needs a force $\vec{F}_c = -\frac{\Delta x v^2 \mu}{R}\hat{r}$ to go around the circle. The question is, what force is available to make this happen. Let us make Δx big and draw forces:

Immediately, we see that the net force Along x (parallel to string) is zero. But the y-components due to the tension add



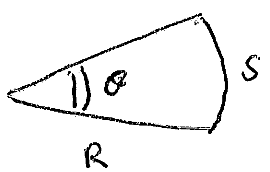
So available force at P is
$$\underset{\rightarrow}{F_A} = -2F_y\,\hat{y}$$
$$= -2F\,Sin\,\theta\,\hat{y}$$
$$\cong -2F\,\theta\,\hat{y} \qquad \text{Since } \theta << 1.$$
$$= -2F\frac{\Delta x}{2R}\hat{y} \qquad \theta = \frac{\Delta x}{2R}$$
$$= -F\frac{\Delta x}{R}\hat{y}$$

While at P the needed $\underset{\rightarrow}{F_c}$ is $-\dfrac{\mu\Delta x\, v^2}{R}\hat{y}$. If $\underset{\rightarrow}{F_A} = \underset{\rightarrow}{F_c}$ Δx can happily participate in the pulse. That is, we must require $\dfrac{F\Delta x}{R} = \dfrac{\mu\Delta x v^2}{R}$. So $v = \sqrt{\dfrac{F}{\mu}}$ is the speed of a small amplitude pulse in a string which has a tension F in it and a linear density (mass per unit length) of $\mu\ kg/m$. It seems reasonable that for a periodic wave on our string we can write

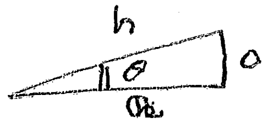
$$y = A\,Sin(kx - \omega t)$$
$$\omega = vk$$
$$v = \sqrt{\frac{F}{\mu}}$$

Provided that we keep $A << \lambda$ so all angles are small [we needed $\theta << 1$ in our proof].

[Note that when $\theta << 1$, $Sin\,\theta \approx \theta$



$\theta = \dfrac{S}{R}$



$Sin\,\theta = \dfrac{o}{h}$

$\theta << 1$

$a \to h$

$\dfrac{o}{a} \approx \dfrac{o}{h} \approx \dfrac{S}{R}$]

ENERGY TRANSPORT BY SINE WAVE ON STRING

Every point on the string has a *y coordinate* which varies as $y = A\,Sin\,\omega t$. This is like linear harmonic motion so every point has a transverse velocity

v_y ~~vy~~ $= A\omega\ Cos\,\omega t$

A unit length of string will therefore have a kinetic energy $K = \frac{1}{2}\mu A^2\omega^2\cos^2\omega t$

Whose maximum value (which is equal to total energy, KIN +POTL) * will be $K_{max} = \frac{1}{2}\mu A^2\omega^2$

wave travels by *v m/s* so energy transport per second $\eta = \frac{1}{2}\mu A^2\omega^2 v$

Since $F = \mu v^2$ we can also write $\eta = \frac{1}{2}A^2\omega^2\frac{F}{v}$

* TREAT A UNIT LENGHTH OF STRING AS A "SPRING-MASS" OSCILLATOR with spring constant k_0.

KIN. ENERGY $K = \frac{1}{2}\mu A^2\omega^2 Cos^2\omega t$

POT. ENERGY $U = \frac{1}{2}k_0 A^2 Sin^2\omega t$

But $\omega = \sqrt{\frac{k_0}{\mu}}$

So $U = \frac{1}{2}\mu A^2\omega^2 Sin^2\omega t$

Next, averaged over time $\langle Sin^2\omega t\rangle = \langle Cos^2\omega t\rangle = \frac{1}{2}$

So $\langle K\rangle + \langle U\rangle = \left(\frac{1}{4}+\frac{1}{4}\right)\mu A^2\omega^2$

$= \frac{1}{2}\mu A^2\omega^2$

[of course, $Sin^2\omega t + Cos^2\omega t = 1$ so it is not surprising that $\langle K\rangle + \langle U\rangle = K + U = K_{max} = U_{max}$ since our oscillator has no friction, total energy is CONSTANT!]