**Homework #5** — Phys625 — Spring 2002
**Deadline: Tuesday, March 12, 2002.**
Turn in homework in the class or put it in the box on the door of Phys 2314 by 10 a.m.

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**Do not forget to write your name and the homework number!**
Equation numbers with the period, like (3.25), refer to the equations of the textbook.
Equation numbers without period, like (5), refer to the equations of this homework.

# Polarization Operator (read the rest of Ch. II)

**1.** *General expression* [**4 points**]

Let us define the density-density correlation function for a Fermi gas as

$$\Pi(\mathbf{r}-\mathbf{r}', t-t') = i\langle \mathrm{T}\,\hat{n}(\mathbf{r},t)\,\hat{n}(\mathbf{r}',t')\rangle. \tag{1}$$

Derive a general expression for the density-density correlator (1) in the momentum representation, $\Pi(\Omega,\mathbf{q})$.

In order to do that, draw the Feynman diagram that corresponds to the calculation of (1) and express $\Pi(\Omega,\mathbf{q})$ in terms Green's functions of the noninteracting electrons, $G_0(\omega,\mathbf{p}) = 1/[\omega - \epsilon_\mathbf{p} + \mu + i0\,\mathrm{sgn}(\epsilon_\mathbf{p}-\mu)]$. Integrating over the intermediate frequency of the loop, obtain the following expression

$$\Pi(\Omega,\mathbf{q}) = -2\int \frac{d^3p}{(2\pi)^3}\,\frac{f(\epsilon_{\mathbf{p}+\mathbf{q}}) - f(\epsilon_\mathbf{p})}{\Omega - \epsilon_{\mathbf{p}+\mathbf{q}} + \epsilon_\mathbf{p} + i0\,\mathrm{sgn}(\epsilon_{\mathbf{p}+\mathbf{q}}-\epsilon_\mathbf{p})}, \tag{2}$$

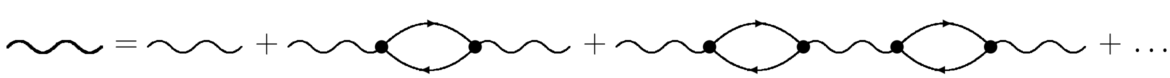
where $f(\epsilon) = \Theta(\mu-\epsilon)$ is the Fermi distribution function at $T=0$.

**2.** *3D and 1D Fermi gases*

Calculate the integral (2) for small $\Omega$ and $q$ ($\Omega, v_F q \ll E_F = \mu$) using the approximate dispersion relation for electrons $\epsilon_p = v_F(p-p_F)$. Do the calculations for 3D [**4 points**] and 1D [**4 points**]. In each case, be sure to calculate both real and imaginary parts. In 3D, the result is called the Lindhard function.

**3.** [**6 points**] *Renormalization of phonons*

Calculate the renormalized Green's function of phonons $D(\Omega,\mathbf{q})$ by summing up the infinite series of diagrams shown below.



Here the wavy lines represent the phonon Green's functions, the bare one $D_0$ and the renormalized one $D$, shown by thin and thick lines. The lines with the arrows represent the bare electron Green functions, and the dots the electron-phonon interaction constant. For the electron-hole loop, use the result of Problem 2 for 3D case.

Determine the renormalized phonon spectrum and phonon lifetime. When doing the calculation, assume that phonons are acoustic, and make the appropriate approximations taking into account that sound velocity is much smaller than the Fermi velocity.

4. **[8 points]** *Plasmons*

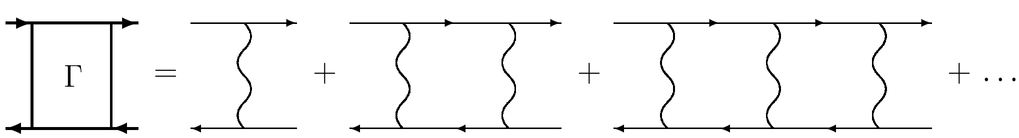
   Now let us discuss renormalization of Coulomb interactions by electron-hole loops. It is represented by the figure in Problem 3, where the thin and thick wavy lines now represent the bare Coulomb interaction $U_0(\mathbf{q}) = 4\pi e^2/q^2$ and the renormalized Coulomb interaction $U(\Omega, \mathbf{q})$.

   Using the result of Problem 2 for the electron-hole loop in the 3D case, calculate $U(\Omega, \mathbf{q})$. Examine the poles of $U(\Omega, \mathbf{q})$ as a function of $\Omega$ for small $\mathbf{q}$ and find the plasma oscillation frequency.

   Also consider $U(0, \mathbf{q})$ in the static limit $\Omega = 0$. Fourier transform $U(0, \mathbf{q})$ from the momentum space to the real space (assuming that only small $q$ matter) and determine the screening length. Compare the screening length with the average distance between electrons.

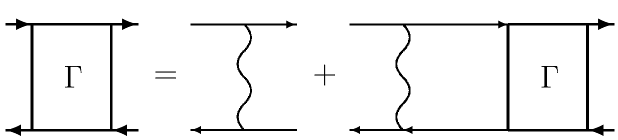
5. **[6 points]** *Zero-sound*

   Consider the following infinite series of diagrams for the two-particle Green's function $\Gamma$ of electrons




   Here the wavy lines represent an interaction between electrons $U(\mathbf{q})$.

   Show that this geometrical progression can be written as




   Write the analytical expression for this diagram. Let us consider it in the limit where the total frequency $\Omega$ and $\mathbf{q}$ in the electron-hole channel are small. Assuming that $\Gamma$ does not depend significantly on the intermediate frequency of the electron-hole loop, the integral over that frequency can be taken explicitly, and you can use expression (2) for the electron-hole loop. In this way, obtain an integral equation for $\Gamma$ with the integration over the intermediate momentum of the electron-hole loop.

   Consider singularities of $\Gamma$ as a function of the total frequency $\Omega$, which give the spectrum of two-particle, electron-hole excitations. Proceeding in a manner somewhat similar to Problem 4b of HW3, obtain the integral equation for the eigenmodes of zero sound, and show that it is the same as discussed in the Fermi liquid theory. Hint: Basically, you need to derive Eq. (18.1), with some obvious change of notation.