Homework #4 — Phys625 — Spring 2002
Victor Yakovenko, Associate Professor
Office: Physics 2314
Phone: (301)-405-6151
E-mail: yakovenk@physics.umd.edu
Deadline: Thursday, February 28, 2002.
Turn in homework in the class or put it in the box on the door of Phys 2314 by 10 a.m.
Web page: http://www2.physics.umd.edu/~yakovenk/teaching/phys625.spring2002

Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of the textbook.
Equation numbers without period, like (5), refer to the equations of this homework.

Electron-phonon interaction (§§12, 13, 14, 64, 65)

1. *Green's function of phonons*

(a) [**2 points**] In Problem 2 of HW 1, we introduced the second-quantized operator of lattice displacements:

$$\hat{\mathbf{u}}(\mathbf{r},t) = \sum_{\mathbf{k},\alpha} \mathbf{e}_{\mathbf{k},\alpha} \sqrt{\frac{\hbar}{2V\rho\omega_{0,\alpha}(\mathbf{k})}} \left(\hat{c}_{\mathbf{k},\alpha} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{0,\alpha}t} + \hat{c}^+_{\mathbf{k},\alpha} e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_{0,\alpha}t}\right). \tag{1}$$

Here we made a number of generalizations:

- The phonon operators are denoted as $\hat{c}$ instead of $\hat{a}$.
- We use the continuous coordinate $x = na$, where the a is the intersite distance, instead of the discrete n. Correspondingly, $\rho = m/a$ is the mass density of the lattice.
- The crystal is 3D, so the coordinate $\mathbf{r}$ and momentum $\mathbf{k}$ are 3D vectors.
- The displacement $\mathbf{u}$ is also a 3D vector, characterized by three polarizations α and polarization vectors $\mathbf{e}_{\mathbf{k},\alpha}$. We will consider only the longitudinal phonons with $\mathbf{e}_{\mathbf{k},l}\|\mathbf{k}$ and ignore the transverse phonons with $\mathbf{e}_{\mathbf{k},t} \perp \mathbf{k}$.
- The time dependence appears because we consider $\hat{u}$ in the Heisenberg representation. Correspondingly, $\omega_{0,\alpha}(\mathbf{k})$ is the frequency of a phonon with momentum $\mathbf{k}$ and polarization α.
- The volume V in denominator provides correct normalization.

Show that the following operator gives the local deviation of the lattice mass density from the equilibrium value:

$$\hat{\rho}'(\mathbf{r},t) = \rho \,\mathrm{div}\, \hat{\mathbf{u}}(\mathbf{r},t) = \rho\frac{\partial \hat{\mathbf{u}}}{\partial \mathbf{r}} = \sum_{\mathbf{k}} ik \sqrt{\frac{\hbar\rho}{2V\omega_{0,\alpha}(\mathbf{k})}} \left(\hat{c}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{0,\alpha}t} - \hat{c}^+_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_{0,\alpha}t}\right). \tag{2}$$

Notice that Eq. (2) is consistent with the definition (24.10) for the acoustic phonons with $\omega_0 = sk$, where s is the sound velocity (denoted by u in the book).

(b) [**4 points**] Green's function for phonons is defined similarly to that of fermions, with no sign change upon permutation of operators:

$$D(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = -i\langle T\hat{\rho}'(\mathbf{r}_1,t_1)\hat{\rho}'(\mathbf{r}_2,t_2)\rangle = -i \begin{cases} \langle\hat{\rho}'(\mathbf{r}_1,t_1)\hat{\rho}'(\mathbf{r}_2,t_2)\rangle, & t_1 > t_2 \\ \langle\hat{\rho}'(\mathbf{r}_2,t_2)\hat{\rho}'(\mathbf{r}_1,t_1)\rangle, & t_1 < t_2, \end{cases} \tag{3}$$

where the averaging is done over the ground state with no phonons present. Using the definition (2), calculate the phonon Green's function (3).

Electron-phonon interaction is given by Eq. (64.2), which is consistent with the result of Problem 4 of HW in the limit of long-wavelength acoustic phonons. The rules of diagram technique for electron-phonon interaction are described in §64. In the following four problems you are asked to calculate the electron self-energy function $\Sigma(\omega, \mathbf{p})$ in 3D due to interaction with acoustic phonons, which is given by the diagram (64.5) and Eq. (65.2).

2. [8 points] *Polaron in weak-coupling limit*

Calculate $\Sigma(\omega, \mathbf{p})$ for a single electron interacting with phonons. The single electron (as opposed to electron gas) means that the Fermi energy $\mu \to 0$. Thus use the following Green's function for the electron line in the diagram: $G(\omega, \mathbf{p}) = (\omega - p^2/2m + i0)^{-1}$. (Notice the absence of sgnω.)

Consider $\Sigma(\omega, \mathbf{p})$ close the the "mass surface" $\omega = p^2/2m$ and for small momenta $p \ll ms$. Obtain the expansion:

$$\Sigma(\omega, \mathbf{p}) = \epsilon_0 - \alpha_1 \left(\omega - \frac{p^2}{2m} \right) - \alpha_2 \frac{p^2}{2m}. \tag{4}$$

Show that α_2 determines the renormalization of electron mass m, α_1 the renormalization of the coefficient Z, and ϵ_0 gives the polaron binding energy. Find the effective mass of polaron and comment whether it is heavier or lighter than the bare electron mass.

Hint: When doing the integrals over momentum, use the change of integration variables described in the paragraph after Eq. (65.4), but do not assume that the momenta are close to p_F, because there is no Fermi surface in this problem.

3. [8 points] *Cherenkov radiation of sound*

$\Sigma(\omega, \mathbf{p})$ found in Part 2 has an imaginary part when $v = p/m > s$, because a supersonic electron can emit phonons.

Calculate the decay rate of electron due to emission of phonons, $\gamma = -\text{Im}\Sigma$.

Represent ImΣ as $\int W(\theta)\, d\theta$, where θ is the angle between the momenta of phonon and electron, and find the angular distribution function $W(\theta)$ for phonon emission.

Calculate the phonon emission rate per unit time directly using Fermi golden rule for electron-phonon interaction (64.2) and compare the results.

4. [6 points] *Decay rate for electron gas*

Calculate (estimate) the decay rate $-\text{Im}\Sigma(\omega)$ due to interaction with phonons for an electron gas for ω much smaller than the Fermi energy μ. (In this and the next problem, we consider the case $\mu \neq 0$.)

5. [6 points] *Renormalization of electron spectrum*

Calculate (estimate) Re$\Sigma(\omega, \mathbf{p})$ for an electron gas and show that it depends on ω, but practically does not depend on $\mathbf{p}$. Show that Re$\Sigma(\omega)$ is significant only for $\omega \ll \omega_D$, where $\omega_D = sk_D$ is the Debye energy and k_D is the Debye momentum (the upper momentum cutoff for phonons). Show that the electron mass increases for such energies.