Statistical Mechanics of Money

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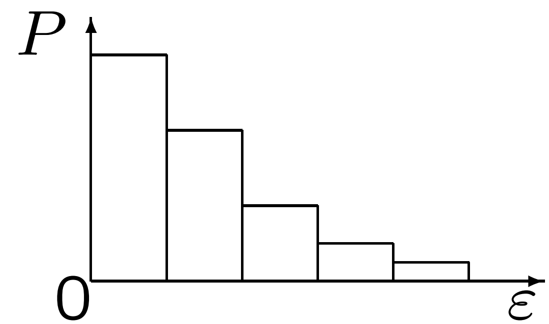
The European Physical Journal B **17**, 723 (2000)
cond-mat/0001432

Derivations of the Boltzmann-Gibbs distribution $P(\varepsilon) \propto \exp(-\varepsilon/T)$ of energy ε in statistical physics:



- As a "stable distribution": $\boxed{\varepsilon_1 \mid \varepsilon_2} = \boxed{\varepsilon}$

  $\varepsilon_1 + \varepsilon_2 = \varepsilon$, $P(\varepsilon_1)P(\varepsilon_2) = P(\varepsilon) \Rightarrow P(\varepsilon) \propto \exp(-\varepsilon/T)$

- By maximizing entropy $S = -\sum_\varepsilon P(\varepsilon) \ln P(\varepsilon)$ under the constraint of conservation law: $\sum_\varepsilon P(\varepsilon)\, \varepsilon = \text{const}$.

- As a stationary solution of the Boltzmann equation:

$$dP(\varepsilon)/dt = \sum_{\varepsilon',\,\Delta} -w_{[\varepsilon,\varepsilon']\to[\varepsilon+\Delta,\varepsilon'-\Delta]}\, P(\varepsilon)P(\varepsilon') + w_{[\varepsilon+\Delta,\varepsilon'-\Delta]\to[\varepsilon,\varepsilon']}\, P(\varepsilon+\Delta)P(\varepsilon'-\Delta).$$

  If $w_{[\varepsilon,\varepsilon']\to[\varepsilon+\Delta,\varepsilon'-\Delta]} = w_{[\varepsilon+\Delta,\varepsilon'-\Delta]\to[\varepsilon,\varepsilon']}$ (time-reversal symmetry), then $P(\varepsilon) \propto e^{-\varepsilon/T}$ is stationary: $\frac{dP(\varepsilon)}{dt} = 0$.

Analogy between Economics and Statistical Mechanics

	Economics	Statistical Mechanics
System	Agents	Particles
Conserved quantity	Money, m	Energy, ϵ
Probability distribution	$P(m)$ - ?	Gibbs law, $P(\epsilon) \propto e^{-\epsilon/T}$

As energy in physics, money is conserved in each economic transaction. In trading, money is only transferred from one agent to another, not created or destroyed. Because of the conservation law, by analogy with classical statistical mechanics, we expect that the probability distribution of money, $P(m)$, obeys: $P(m_1 + m_2) = P(m_1)P(m_2)$. Thus, $P(m)$ must be the exponential Gibbs distribution $P(m) \propto e^{-m/T}$, where T is the effective temperature of the economic system (expressed in dollars, for example).

Money, Wealth, and Income

Wealth $=$ Money $+$ Property (material wealth)

Money is conserved.

Material wealth is not conserved.

d(Money)/dt $=$ Income - Spending

Modeling

- Many agents, $N \gg 1$. Agent j has the amount of money $m_j \geq 0$. Initially, all agents are equal: $m_i = M/N$, $\forall i$.
- Infinitely long range interaction. Any agent can interact with any other agent with equal probability.
- Randomly chosen agents interact pairwise (one pair at a time) and exchange money $\Delta m > 0$:
$$[m_i, m_j] \longrightarrow [m_i', m_j'] = [m_i + \Delta m, m_j - \Delta m].$$
If $m_j' < 0$, the transaction does not take place.
- The agent i whose money increases $(m_i' > m_i)$ is called a winner. The agent j whose money decreases $(m_j' < m_j)$ is called a looser.

- Money is conserved by interactions:

$$m_i + m_j = m_i' + m_j', \qquad \sum_{i=1}^{N} m_i = M = \text{constant}.$$

- After many transactions, we find the stationary probability distribution of money $P(m)$ shown in Fig. 1. The stationary distribution does not depend on the choice of trading rules (described below as Models 1, 2, 3, A, B, Firms). The Gibbs law $P(m) \propto e^{-m/T}$ fits the stationary distribution with $T = M/N =$ average money per agent.

- Entropy $S = -\sum_m P(m) \ln P(m)$ is shown in Fig. 2 as a function of time. Entropy increases in time and saturates for the stationary distribution. Entropy is maximal for the Gibbs distribution.

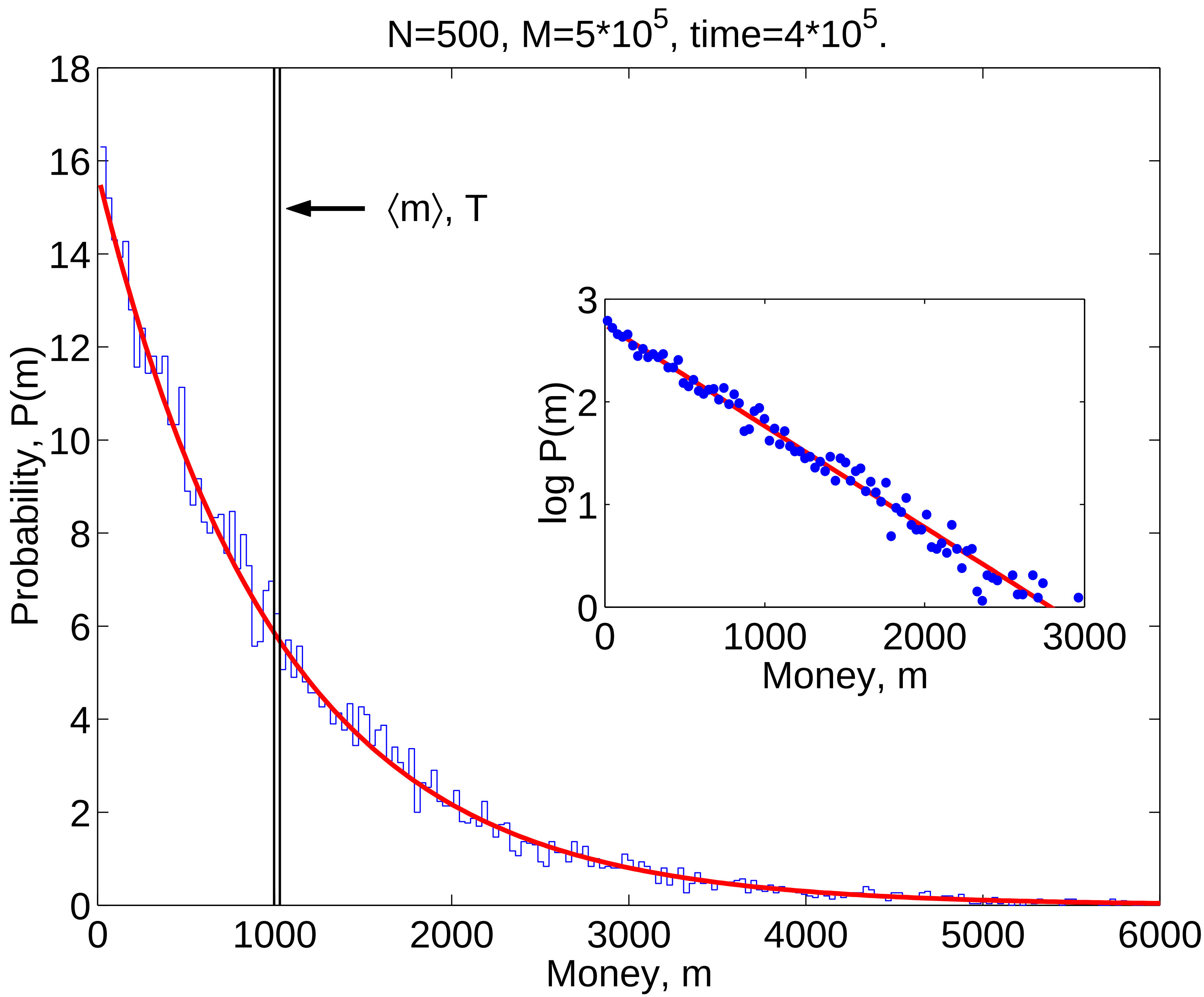


Fig. 1. Stationary probability distribution of money $P(m)$ in Model 3. The solid line is a fit to the Gibbs law $P(m) \propto \exp(-m/T)$.

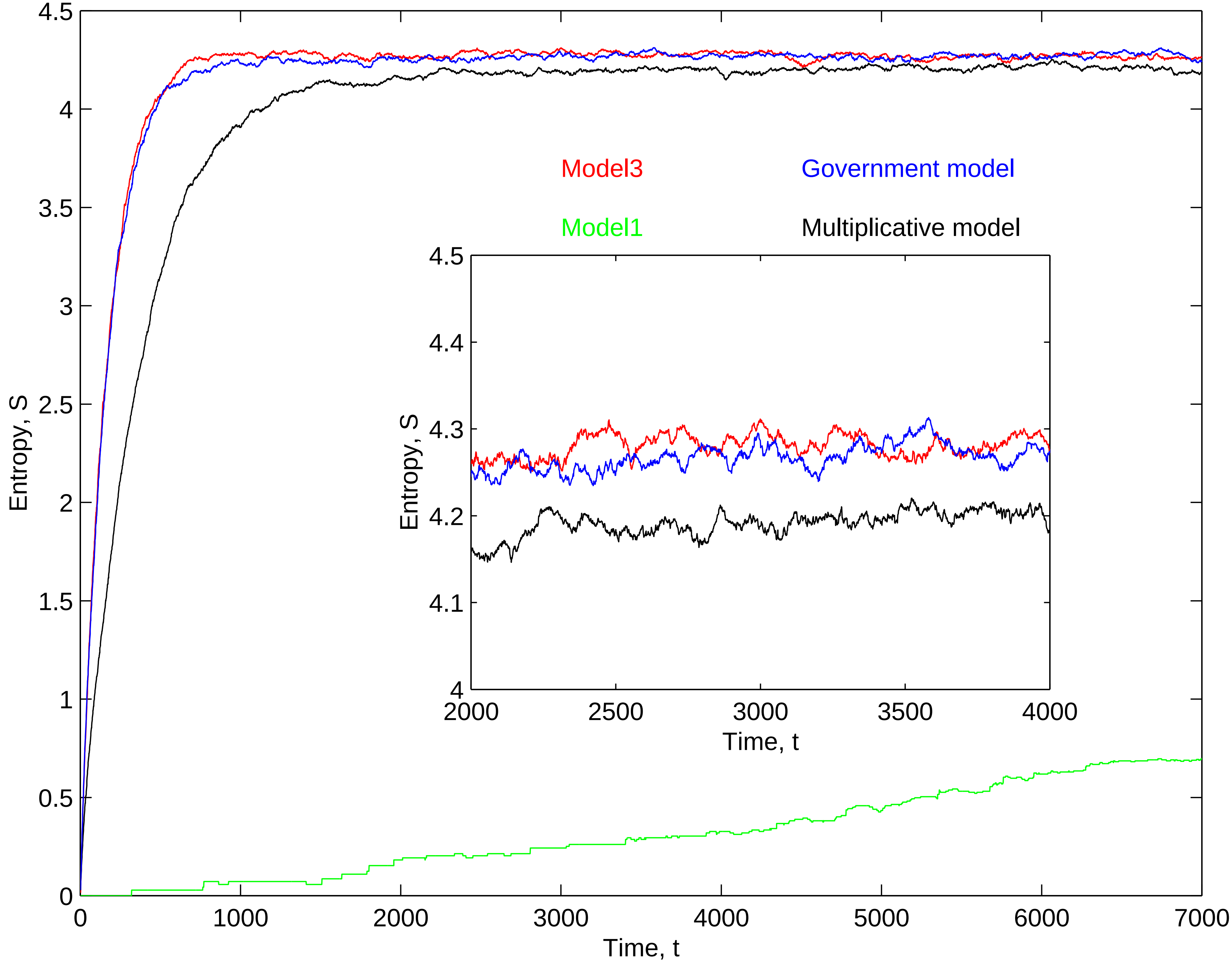


Fig. 2. Entropy as a function of time for Model 1, Model 3, Government model, and Multiplicative model.

Trading Rules

Model 1. Exchange a small constant amount Δm, say \$1.

Model 2. Exchange a random fraction $0 \leq \alpha \leq 1$ of the average wealth of the pair: $\Delta m = \alpha(m_i + m_j)/2$.

Model 3. Exchange a random fraction of the average wealth of the economy: $\Delta m = \alpha\, M/N$.

Selection of Winners and Losers

Model A. For a given pair (i, j), winner and loser are selected randomly every time they interact: $i \longleftrightarrow j \longleftrightarrow k$.

Model B. For every pair, winner and loser are randomly established once, before the interactions start. In this case, money flows along directed links between the agents: $i \longrightarrow j \longrightarrow k$.

Model with Firms

To better simulate economy, we introduce firms in the model.

- One agent at a time is randomly selected to be a "firm":
  - borrows capital K from a randomly selected agent and then returns it with an interest rK
  - hires L other randomly selected agents and pays them wages W
  - makes Q items of a product and sells it to Q randomly selected agents at a price R
- The net result is a many-body interaction, where
  - one agent increases his money by rK
  - L agents increase their money by W
  - Q agents decrease their money by R
  - the firm receives profit $\pi_f = RQ - LW - rK$

- The parameters of the model are selected following the procedure described in economics textbooks:
  - The aggregate demand-supply curve for the product is taken to be: $R(Q) = V/Q^\alpha$, where Q is the quantity people would buy at a price R, and $\alpha = 0.5$ and $V = 100$ are constants.
  - The production function of the firm has the conventional Cobb-Douglas form: $Q(L,K) = L^\beta K^{1-\beta}$, where $\beta = 0.8$ is a constant.
  - In our simulation, we set $W = 10$. After maximizing π_f with respect to K and L, we find: $L = 20$, $Q = 10$, $R = 32$, $\pi_f = 68$.
- The stationary probability distribution of money in this model again has the Gibbs form $P(m) \propto \exp(-m/T)$.

Boltzmann Equation

$$\frac{dP(m)}{dt} = \sum_{m',\Delta} -w_{[m,m']\to[m+\Delta,m'-\Delta]}\, P(m)P(m') + w_{[m+\Delta,m'-\Delta]\to[m,m']}\, P(m+\Delta)P(m'-\Delta),$$

where $w_{[m,m']\to[m+\Delta,m'-\Delta]}$ is the probability for the trade $[m,m'] \to [m+\Delta, m'-\Delta]$ to happen. If the model has the time-reversal symmetry, then

$$w_{[m,m']\to[m+\Delta,m'-\Delta]} = w_{[m+\Delta,m'-\Delta]\to[m,m']}.$$

In this case, the Gibbs distribution $P(m) \propto \exp(-m/T)$ is stationary: $dP(m)/dt = 0$. However, if the time-reversal symmetry is broken, the system may have a non-Gibbs stationary distribution or no stationary distribution at all.

Boltzmann Equation for Model 1 ($1 exchange)

$$\begin{aligned}\frac{dP_m}{dt} &= P_{m+1}\sum_{n=0}^{\infty}P_n + P_{m-1}\sum_{n=1}^{\infty}P_n \\ &\quad -P_m\sum_{n=0}^{\infty}P_n - P_m\sum_{n=1}^{\infty}P_n \\ &= (P_{m+1}+P_{m-1}-2P_m) + P_0(P_m - P_{m-1}),\end{aligned}$$

where $\sum_{n=0}^{\infty}P_n = 1$.

The stationary solution is $P_m = e^{-m/T}(1-e^{-1/T})$

Models with non-Gibbs Distributions of Money

Multiplicative exchange. The agents exchange a fixed fraction α of looser's money:

$$[m_i,\, m_j] \longrightarrow [(1-\alpha)m_i,\, m_j + \alpha m_i].$$

This model breaks the time-reversal symmetry, because the reversed interaction does not lead to the original configuration: $[m_i,\, m_j] \longrightarrow [(1-\alpha)m_i,\, m_j + \alpha m_i] \longrightarrow [(1-\alpha)m_i + \alpha(m_j + \alpha m_i),\, (1-\alpha)(m_j + \alpha m_i)] \neq [m_i,\, m_j]$

The model was studied by S. Ispolatov, P. L. Krapivsky, S. Redner, Eur. Phys. J. B **2**, 267 (1998), who found non-Gibbs stationary distributions for $\alpha \neq 1/2$. We confirm their result in our simulation shown in Fig. 4.

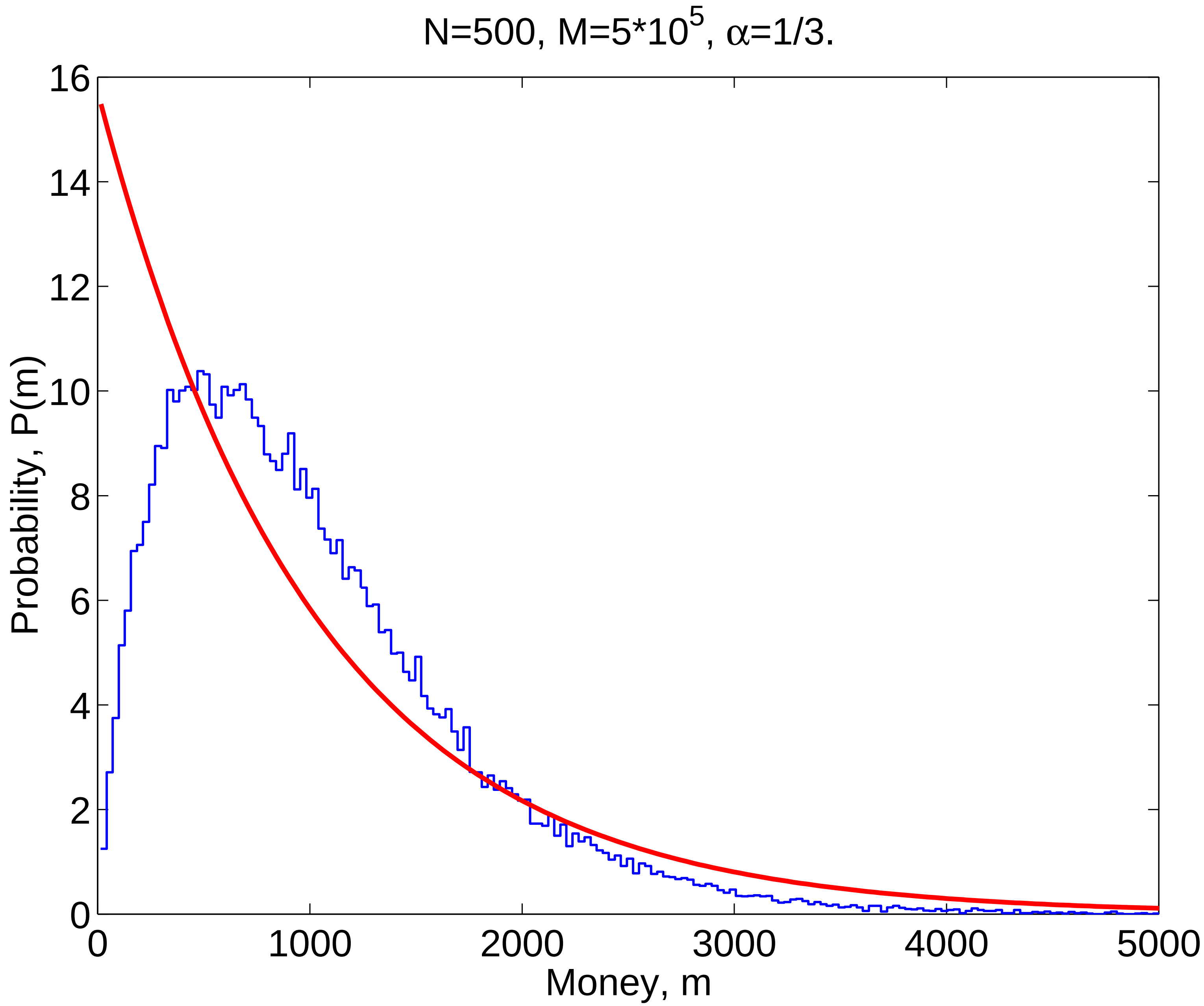


Fig. 4. Stationary probability distribution of money $P(m)$ in the Multiplicative model with $\alpha = 1/3$. The high-m tail of the distribution is exponential.

Taxes and Subsidies. Consider a special agent ("government") that collects a tax on every transaction in the system. The collected money is equally divided between all agents of the system, so that each agent receives the subsidy δm with the frequency $1/\tau_s$. Assuming that δm is small and approximating the collision integral with a relaxation time τ_r, we get a Boltzmann equation

$$\frac{\partial P(m)}{\partial t} + \frac{\delta m}{\tau_s}\frac{\partial P(m)}{\partial m} = -\frac{P(m) - \tilde{P}(m)}{\tau_r},$$

where $\tilde{P}(m)$ is the equilibrium Gibbs function.

The second term acts as an electric force applied to electrons in metal and pumps out the low-money population.

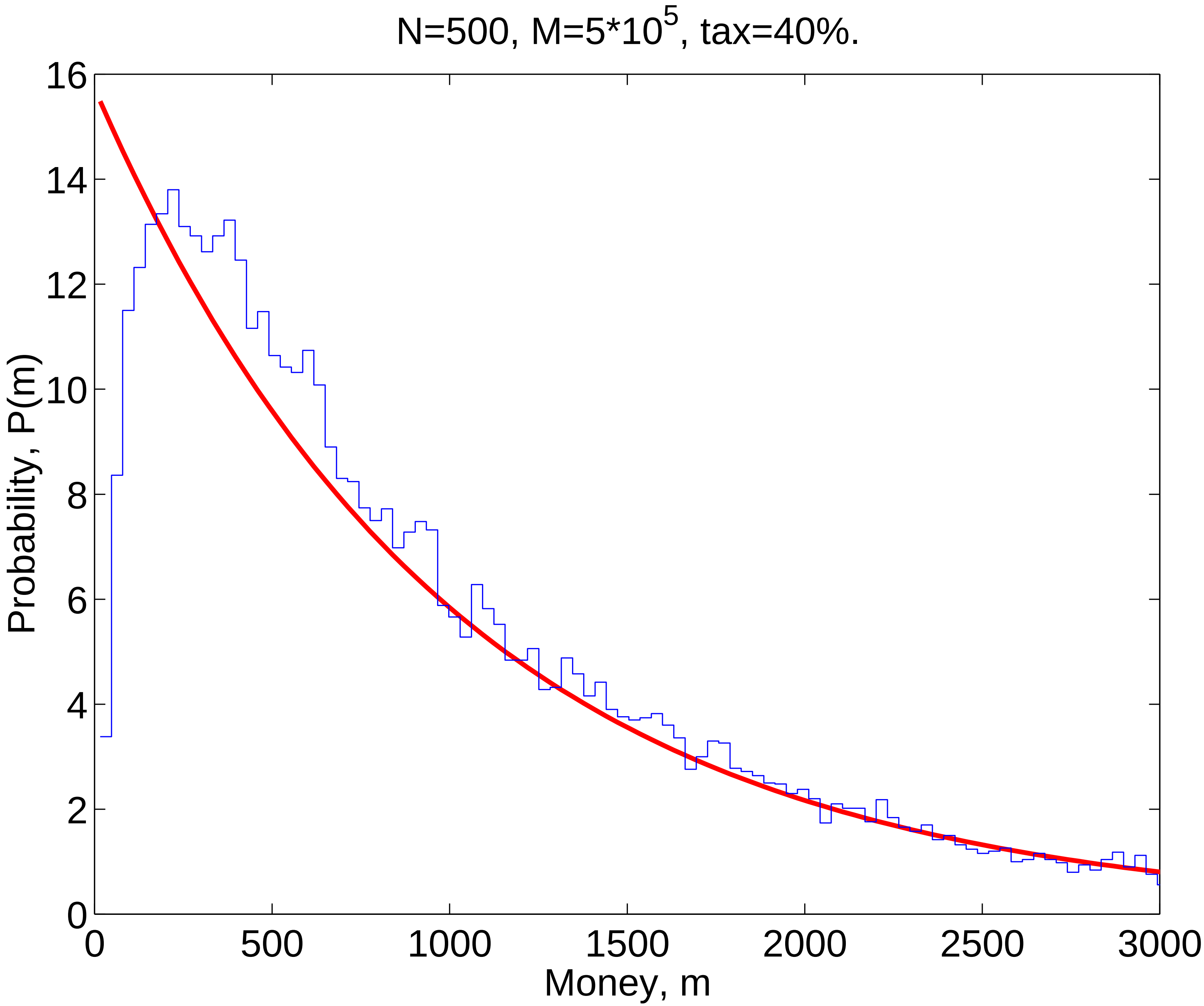


Fig. 3. Histogram: stationary probability distribution of money in the model with taxation and subsidies; solid curve: the Gibbs law.

Models with Debt

- The boundary condition $m \geq 0$ was crucial in establishing the Gibbs distribution law. What changes when agents are permitted to go into debt?
- Now agent's money can be negative, but no lower than a maximum debt, $m > -m_D = -800$.
- The resultant distribution is the Gibbs one again with a higher temperature (Fig. 5). The distribution is broader, which implies higher inequality between agents.

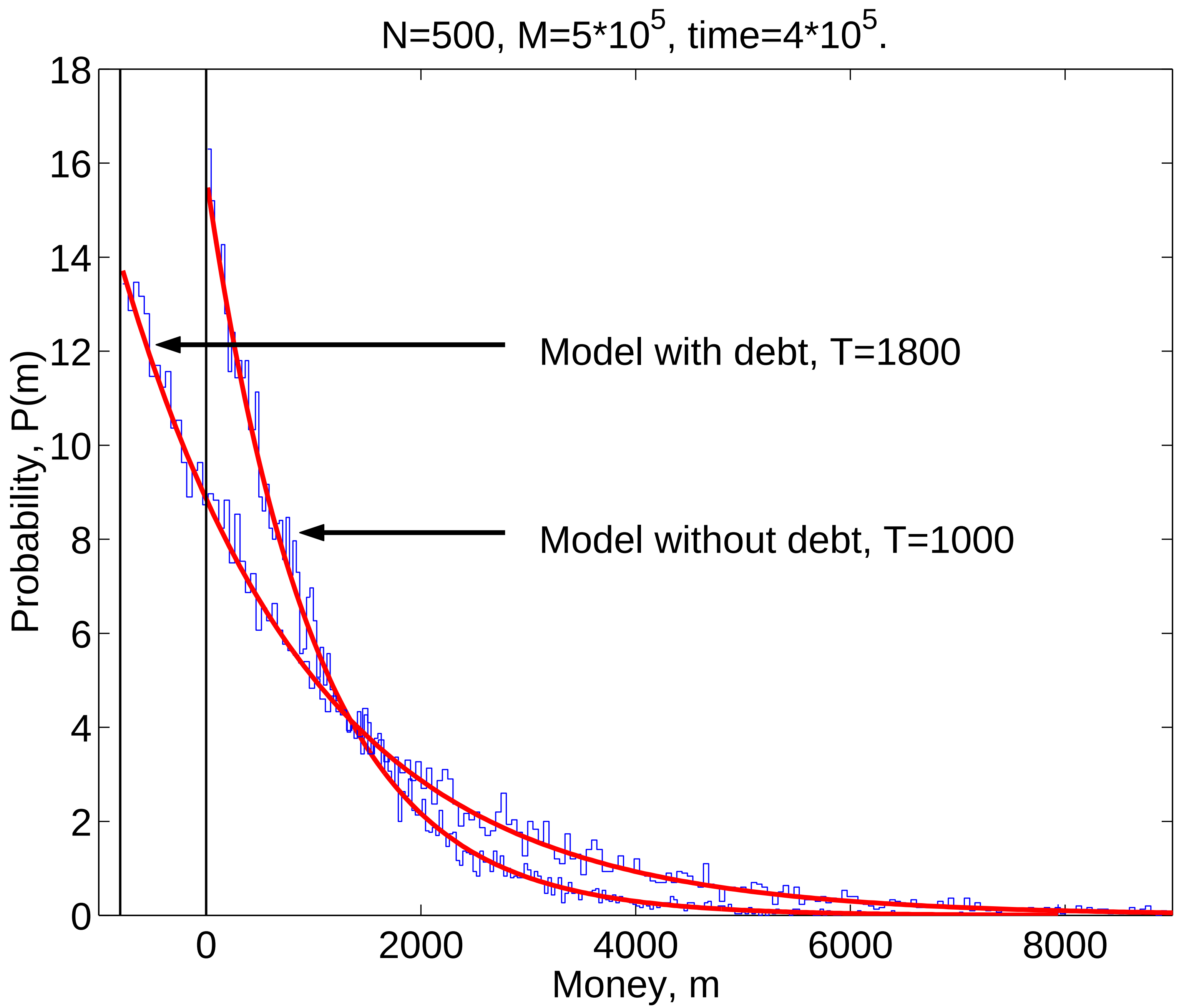


Fig. 5. The stationary probability distributions of money $P(m)$ with and without debt. The solid lines are fits to the Gibbs laws with different temperatures.

Model with Bank

Agents	**Bank**
Trade between themselves	Keeps agents' money as deposits
Get loans from bank if they need	Gives loans until legally allowed
Pay monthly interest if in debt	Pays interest on deposits
Default if unable to pay the debt	Takes the loss on unpaid loans

Time	Depositor		Bank		Borrower	
	Assets	Liablties	Assets	Liabilities	Assets	Liablties
(1)	m					
(2)	m (IOU Bank)		m	m (to Depositor)		
(3)	m (IOU Bank)		m (IOU Borrower)	m (to Depositor)	m	m (to Bank)

Loans and Debts

- When an agent takes a loan, his account is credited with the loan, but also his debt is increased by the value of the loan. We view this as a pair creation of positive (asset) and negative (liability) money. This process conserves the total amount of money.

- The bank that lends money receives an IOU from the borrower, so bank's balance does not change. If a borrower defaults on the loan, his liability is transferred to the bank and annihilates his IOU.

- As money, we count all financial instruments with fixed denomination: currency, IOUs, bonds, etc. We do not consider material assets or stocks as money, because their denomination is not fixed.

- **Net Worth = Assets − Liabilities**

  satisfies the conservation law.

- Traditionally, only assets are considered as money. This quantity is not conserved because of debt. However, if debt changes slowly, assets have a quasi-equilibrium distribution.

- In our model, we consider only one bank and set fixed interest rates for loans (iL) and deposits (iD). We find that bank's activity can evolve into only two extremes:

  (a) The bank gets deeper and deeper into debt, "creating" money in the system

  (b) The bank ends up accumulating all the money in the system

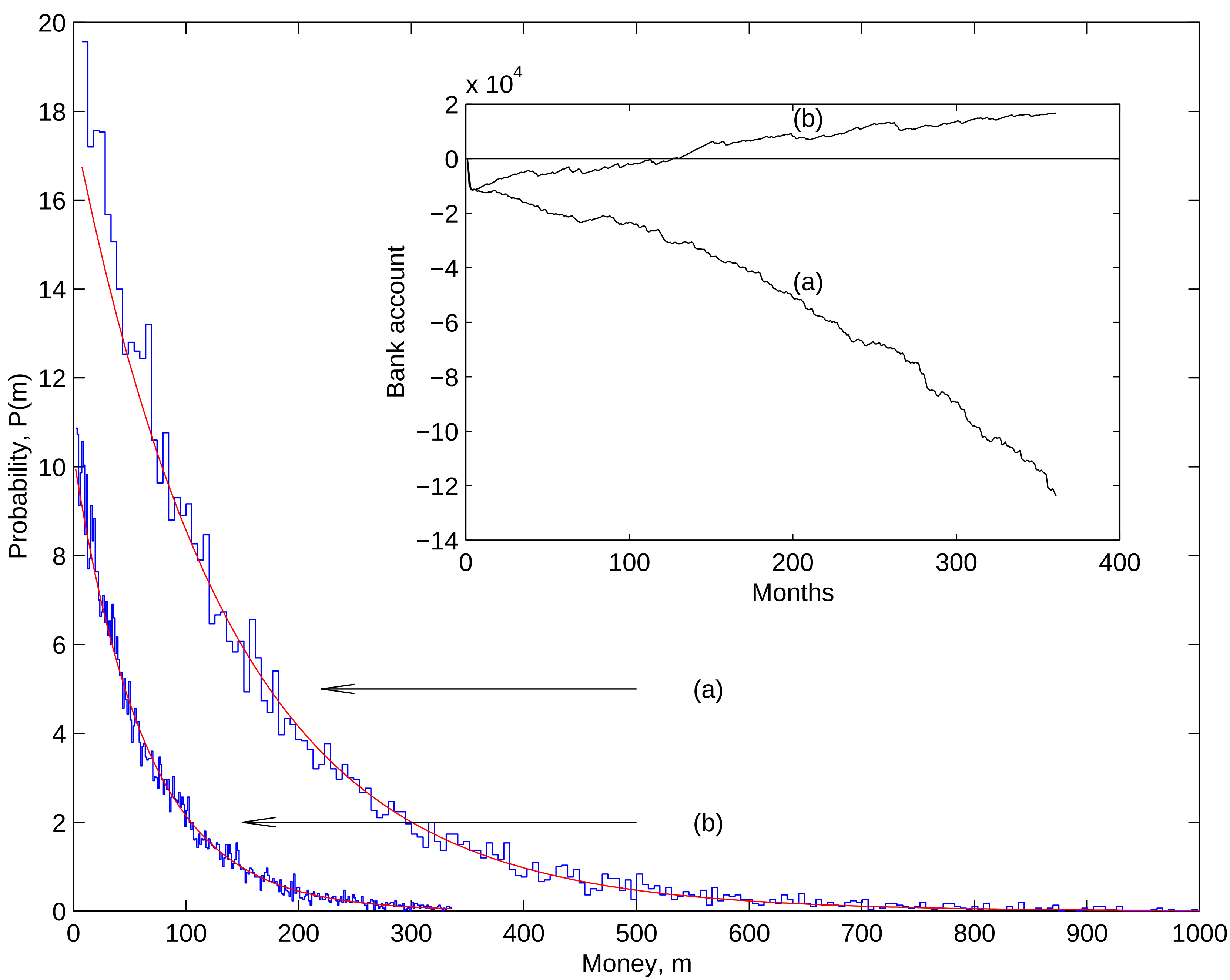


Fig. 6. Probability distributions of assets $P(m)$ in the system interacting with a bank, and the time dependence of the bank account. Trades/Month = (a) 600, (b) 1500.

CONCLUSIONS

In a wide variety of models we find that the stationary probability distribution of money has the exponential Gibbs form

$$P(m) \propto \exp(-m/T)$$

This is a consequence of the conservation law for money.

Deviations from the Gibbs law are found in the models where the time-reversal symmetry is broken.

Perspectives

- Thermal machines: $\boxed{T_1} \longleftarrow \bigcirc \longleftarrow \boxed{T_2}$ $\qquad T_1 < T_2$
- Negative specific heat $dS/dT < 0$: Instabilities (compare with gravitational systems, black holes)
- Experimental check: Comparison with actual economic data
- Aggregation (bound states): Atoms vs molecules
- Phase transitions
- Stock market models: The role of total money flux