1. **Equation of motion for a stretched string**: In class we derived the equation of motion for a stretched string by applying Newton's law to each bit of string. This is also called the one-dimensional wave equation. [4+3+3 +(3+3+4)=20 pts.]

Solution:

(a) The dimensions are tension $[\tau] = MLT^{-2}$, linear mass density $[\rho] = ML^{-1}$ and wavelength $[\lambda] = L$. No dimensionless combination exists, since only the tension involves time, so it must be omitted from any dimensionless quantity, but then only the mass density involves mass, so it too must be omitted, which leaves only the wavelength, which is not dimensionless. The unique combination with dimensions of velocity must involve the ratio τ/ρ, to cancel the M's. In fact $[\tau/\rho] = L^2T^{-2}$, so the velocity is proportional to $\sqrt{\tau/\rho}$.

(b) If the string has fixed endpoints, then the problem depends also on the string length l. The lowest frequency normal mode must have frequency proportional to $(\sqrt{\tau/\rho})/l$.

(c) Using the chain rule, $\partial_t f(x - vt) = -vf'(x - vt) = -v\partial_x f(x - vt)$, where the notation ∂_t denotes $\partial/\partial t$, and f' indicates derivative of f with respect to its argument. Taking another derivative with respect to t and using $\partial_t\partial_x = \partial_x\partial_t$ yields $\partial_t^2 f = (-v)^2\partial_x^2 f = v^2\partial_x^2 f$.

(d) (i) Inserting $y(x, t) = \sin(\omega t + \varphi)f(x)$ in the wave equation yields

$$f''(x) = -(\omega/v)^2 f(x),$$

where the prime denotes derivative with respect to x. (ii) Being a second order equation the general solution has two free parameters. You could find this by assuming an exponential for $e^{\alpha x}$ and solving for α (there are two imaginary solutions) and then requiring that the solution be real. However, you hopefully remember that the functions that are proportional to the negative of their second derivative are sin and cos. We can write the general solution as is $f(x) = A\sin(\omega x/v+\delta)$, where A and δ are real constants. Then $y(x, t) = \sin(\omega t+\varphi)f(x)$. (iii) The boundary condition $y(0, t) = 0$ implies $\sin\delta = 0$, so $\delta = 0, \pi$. The latter is the same as the former together with a sign flip of A. The boundary condition $y(\ell, t) = 0$ then implies $\sin(\omega\ell/v) = 0$, i.e. $\omega\ell/v = n\pi$ for any integer n. That is, the frequency must have one of the discrete values $\omega_n = n\pi v/\ell$, and the corresponding mode function is $f_n(x) = A_n \sin(n\pi x/\ell)$. The lowest frequency is $\omega_0 = \pi v/\ell$. (Since $v = \sqrt{\tau/\rho}$, this agrees with the dimensional analysis. The dimensionless coefficient is π.)

2. **Convergence of improper integrals**

   (a) Show that $\int_1^\infty dt\, t^n$ is finite if and only if $n < -1$.

   (b) Show that $\int_0^\infty dt (a + bt)^n$, with $a, b > 0$, is finite if and only if $n < -1$.

   Be careful to treat the $n = -1$ cases properly. [5 pts.]

   Solution: If $n \geq 0$ the integrals both obviously diverge: the integrand is constant ($n = 0$) or monotonically increasing ($n > 0$) and the range is infinite. What about negative n? Let's just do the integrals. Replacing the upper limit with t_0, the first integral yields $(n+1)^{-1}(t_0^{n+1} - 1)$, as long as $n \neq -1$. If $n = -1$ the integral is $\ln t_0$. The result is therefore finite as $t_0 \to \infty$ if and only if $n + 1 < 0$, i.e. $n < -1$. The argument for the second integral is identical.

3. Consider a particle of mass m in one dimension with a positive velocity v, acted on by a force that depends on the velocity as $-bv^n$, where b is a positive constant and n is a positive dimensionless number. This force acts to slow the particle down.

   (a) Use dimensional analysis to find an expression for how (i) the time for the particle to come to rest, and (ii) the distance it travels before coming to rest, can depend on the initial velocity v_0, together with m, b, and n. [5 pts.]

   Solution: $F = ma$ becomes here $-bv^n = mdv/dt$. Solving for dt yields $dt = -(m/b)v^{-n}dv$. Therefore to make a time to stop t_{stop} from the available quantities we can write

   $$t_{\text{stop}} = f(n)(m/b)v_0^{1-n}, \tag{1}$$

   where $f(n)$ is an arbitrary function of n. Moreover, this is the *only* way to do it, since it can be checked that there is no dimensionless combination of m, b, v_0. Multiplying this by v_0 we obtain a quantity with dimensions of length,

   $$d_{\text{stop}} = g(n)(m/b)v_0^{2-n}, \tag{2}$$

   where the dimensionless function $g(n)$ could of course be different from $f(n)$. These expressions make good sense: it takes longer to stop with more mass m or a smaller drag coefficient b or a greater intitial velocity v_0... wait a minute! Only if $n < 1$ does greater velocity imply greater t_{stop} in (1). This tells us something important: t_{stop} must be *infinite* if $n \geq 1$. Otherwise dimensional analysis would tell us the absurd result that it takes les time to stop when the initial velocity is higher. Similarly d_{stop} must be infinite if $n \geq 2$.

   (b) By integrating Newton's law, determine for which values of n the particle comes to rest in a finite time, and determine that time. Compare with part 3a. [5 pts.]

   Solution: Newton's law gives $v^{-n}dv = -(b/m)dt$, and integrating both sides yields:

   $$\int_{v_0}^{v_f} v^{-n}\, dv = -(b/m) \int_0^{t_f} dt. \tag{3}$$

   If $n = 1$ we find

   $$\ln(v_f/v_0) = -(b/m)t_f. \tag{4}$$

The left hand side goes to $-\infty$ as v_f goes to 0, so t_f also goes to ∞, which means it takes an infinite amount of time for v to reach zero. If $n \neq 1$ then we find

$$v_f^{-n+1} - v_0^{-n+1} = -(-n+1)(b/m)t_f. \tag{5}$$

If $n > 1$ again the left hand side diverges as $v_f \to 0$, so $t_f \to \infty$. If $n < 1$ the left hand side is finite for $v_f = 0$, and the time to stop is $t_{\rm stop} = (1-n)^{-1}(m/b)v_0^{1-n}$. This agrees with our dimensional analysis (2), with $f(n) = (1-n)^{-1}$, provided $n < 1$. So what went wrong with the dimensional analysis when $n \geq 1$? Nothing—it's just that the function $f(n)$ is infinite in that case! That is, the dimensional analysis alone could not tell us whether the dimensionless coefficient is finite or infinite.

(c) Determine for which values of n the particle travels a finite total distance before coming to rest (whether or not it actually stops in a finite time). Find an expression for that distance and compare with your result from part 3a. [5 pts.]

Solution: If the time to stop is finite, then clearly the particle goes a finite distance before stopping. This is the case for $n < 1$. But perhaps the distance can be finite even if the time is infinite, provided it slows down quickly enough. The distance is related to the time and velocity by $dx = vdt$. When $n = 1$ (4) yields $v(t) = v_0 e^{-bt/m}$, so the distance traveled is

$$\Delta x = \int_0^\infty v_0 e^{-bt/m}\, dt = mv_0/b \tag{6}$$

which is finite. When $n \neq 1$ (5) yields

$$\Delta x = \int_0^\infty v\, dt \tag{7}$$

$$= \int_0^\infty (v_0^{-n+1} + (n-1)(b/m)t)^{-1/(n-1)}\, dt. \tag{8}$$

The integral converges at the upper limit provided the integrand falls off faster than t^{-1}, as shown in problem 2. That is it converges provided $n < 2$. So, for $n \geq 2$ the particle goes infinitely far in an infinite time, while for $1 \leq n < 2$ the particle only goes a finite total distance, but it takes an infinite time to come to rest, and for $n < 1$ the particle stops after a finite time. To evaluate the distance (7), let's choose units with $m = b = v_0 = 1$. The integral then yields $-1/(n-2)$. In arbitrary units this must be the multiplied by the unique quantity with dimensions of length that can be formed from m, b, v_0, that is we get $\Delta x = mv_0^{2-n}/b(2-n)$. For $n = 1$ this agrees with the result obtained in (6). For $n = 0$ it is equivalent to $b\Delta x = mv_0^2/2$, which equates the work done by the constant force b to the change in kinetic energy.