Homework 4

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1.a The null generators are radial and are specified by $r(\lambda)$ and $t(\lambda)$, with fixed angles. As they form a null geodesic congruence, they extremize the functional $I = \frac{1}{2}\int(\dot{t}^2 - \dot{r}^2)$. (The angular part is trivially stationary since it is quadratic in the derivatives $\dot{\theta} = \dot{\phi} = 0$.) The Euler-Lagrange equations for r following from this functional are $0 = \ddot{r}$. Hence, $r = a\lambda + b$ is an affine parameter for the null generators.

1.b The area of the spherical surfaces $dr = 0$ of radius r are Area $= 4\pi r^2$. Then, on the past light cone[1]

$$\begin{aligned} \theta_0 &= \left.\frac{\frac{d\text{Area}}{dr}}{\text{Area}}\right|_{r=r_0} \\ &= \frac{-2\cdot 4\pi r_0}{4\pi r_0^2} \\ &= -\frac{2}{r_0}\ . \end{aligned} \tag{1}$$

1.c According to the focusing theorem, $\theta \to -\infty$ at or before an affine parameter time $\lambda_0 = 2|\theta_0|^{-1}$. Solving (1) gives $r_0 = -\frac{2}{\theta_0} = 2|\theta_0|^{-1}$. On the cone $0 = t^2 - r^2$ this corresponds to a time $\lambda_0 = t_0 = r_0$ in agreement with the theorem.

2.a By the spherical symmetry of the metric, initially radial geodesics remain radial. Furthermore, the square of the 4-velocity is conserved along a geodesic. Therefore, if the geodesic is radial and null to begin with, it remains radial and null. This is a rigorous argument but perhaps it is helpful to do it explicitly (if not, skip to **2.b**): Geodesics extremize the functional $I = \frac{1}{2}\int\left(f\dot{t}^2 - g\dot{r}^2 - r^2\dot{\Omega}^2\right)d\lambda$, *id est* they satisfy the equations

$$\begin{aligned} \frac{d}{d\lambda}\left(f\dot{t}\right) &= 0 \\ f'\dot{t}^2 - g'\dot{r}^2 + 2\frac{d}{d\lambda}\left(g\dot{r}\right) - 2r\dot{\Omega}^2 &= 0 \end{aligned}$$

[1]The sign comes from the fact that r is decreasing on the past light cone.

$$r^2\ddot{\Omega} \;=\; 0\;. \qquad (2)$$

For initially radial geodesics $\dot{\Omega} = 0$ the third equation implies that they stay radial so we can ignore the Ω-terms. The first equation implies that $\dot{t} = c/f$ for some constant c. On the other hand, radial light ($ds^2 = 0$) rays solve $0 = f dt^2 - g dr^2$, *id est*

$$\frac{dr}{dt} \;=\; \pm\sqrt{\frac{f}{g}}\;. \qquad (3)$$

Therefore, $\dot{r} = \pm\sqrt{f/g}\,\dot{t} = \pm c/\sqrt{fg}$ and one can differentiate this again to obtain

$$\ddot{r} \;=\; -\frac{c^2}{2}\left(\frac{f'}{f^2 g} + \frac{g\prime}{f g^2}\right)\;. \qquad (4)$$

If these formulæ solve the second equation in (2), we will have shown that radial null rays are always geodesics. Indeed, substitution yields

$$f'\frac{c^2}{f^2} - g'\frac{c^2}{fg} + 2g'\frac{c^2}{fg} + 2g\left\{-\frac{c^2}{2}\left(\frac{f'}{f^2 g} + \frac{g\prime}{f g^2}\right)\right\} \;\equiv\; 0\;. \qquad (5)$$

2.b

The equation in the text after (3) shows that $\dot{r}$ is constant—i.e. r is linearly related to the affine parameter λ and is thus affine itself—iff $fg = $ constant.

2.c By symmetry $\sigma = 0 = \omega$ on the radial geodesic congruences so that the Raychaudhuri equation reduces to $\dot{\theta} = -\frac{1}{2}\theta^2 + R_{ab}k^a k^b$. On the other hand $\theta = \frac{1}{\text{Area}}\frac{d}{d\lambda}\text{Area} = 2\dot{r}/r$ which gives $\dot{\theta} = 2\ddot{r}/r - \frac{1}{2}\theta^2$. Therefore, $2\ddot{r}/r = R_{ab}k^a k^b$ so $\ddot{r} = 0 \Leftrightarrow R_{ab}k^a k^b = 0$.

2.d

(*i*) The cosmological constant shifts R_{ab} by a term proportional to the metric and hence contributes a term in the Raychaudhuri equation proportional to $g_{ab}k^a k^b = 0$.

(*ii*) A radial electric field $(\vec{E})^r = F_{0r}$. The stress-energy is given by $T_{ab} \propto F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^2$. Its non-trace part reduces to $T_{ab} \propto F_{a0}F_{b0} - F_{ar}F_{br}$. Since k is radial and null, we can normalize it to $k = (1, 1, 0, 0)$ in Schwarzschild-like coordinates. Then $T_{ab}k^a k^b = F_{r0}^2 - F_{0r}^2 \equiv 0$.

Alternatively, we observe that the 1-form $F_{ab}k^b$ has no angular components and is orthogonal to k^a by the anti-symmetry of F. It follows that it is proportional to k_a: There is some constant c such that $F_{ab}k^b = ck_a$. Then $T_{ab}k^a k^b \propto c^2 k_a k^a - \frac{1}{4}k^2 F^2 = 0$.

For the radial magnetic field $(\vec{B})^r = F_{\theta\phi}$ we observe that $T_{ab}k^a k^b$ must vanish because k has no non-radial parts k^θ and/or k^ϕ.

(*iii*) This time the non-trace part of the stress tensor is $T_{ab} = \partial_a\varphi\partial_b\varphi$ and $T_{ab}k^ak^b = (k\cdot\partial\varphi)^2$ which does not vanish for generic φ.

3.a Let $\hat{x} := (d\lambda/dv)x$ for $x = \theta, \sigma, k$. Then

$$\begin{aligned} \frac{d\hat{\theta}}{dv} &= \frac{d\lambda}{dv}\frac{d\theta}{dv} + \frac{d^2\lambda}{dv^2}\theta \\ &= \left(\frac{d\lambda}{dv}\right)^2 \frac{d\theta}{d\lambda} + \frac{d^2\lambda}{dv^2}\left(\frac{d\lambda}{dv}\right)^{-1}\hat{\theta} \\ \Rightarrow \quad \left(\frac{d\lambda}{dv}\right)^2 \frac{d\theta}{d\lambda} &= \frac{d\hat{\theta}}{dv} - \kappa\hat{\theta} \ . \end{aligned} \tag{6}$$

The Raychaudhuri equation is[2]

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{ab}k^ak^b \ . \tag{7}$$

Multiply this equation by $(d\lambda/dv)^2$ and plug in (6) to obtain

$$\frac{d\hat{\theta}}{d\lambda} = \kappa\hat{\theta} - \frac{1}{2}\hat{\theta}^2 - \hat{\sigma}^2 - R_{ab}\hat{k}^a\hat{k}^b \ . \tag{8}$$

3.b Starting with $\Delta A = \int_B \theta d^2A d\lambda = \int_B \hat{\theta} d^2A dv$, substituting the adiabatic solution of (8), $\hat{\theta} \approx (1/\kappa)R_{ab}\hat{k}^a\hat{k}^b$, and using the Einstein equation, we obtain

$$\begin{aligned} \Delta A &= \frac{8\pi}{\kappa}\int_B T_{ab}\hat{k}^a\hat{k}^b d^2A dv \\ &= \frac{4}{T_H}\int_B T_{ab}\hat{k}^a \underbrace{k^b d^2A d\lambda}_{d\Sigma^b} \ , \end{aligned} \tag{9}$$

where we have used $T_H = \kappa/2\pi$. Since $k^a d\lambda = \xi^a dv$ (recall $k^a = (d/d\lambda)^a$ and we are just making a change of variables to Killing time) it follows that $\hat{k}^a = \xi^a$. Defining the "energy" flux[3] across the horizon H as $\delta E_H = \int_H T_{ab}\xi^a d\Sigma^b$, we obtain the desired result

$$\frac{\Delta A}{4} = \frac{\delta E_H}{T_H} \ . \tag{10}$$

4.a Imagine pretty drawing here.

4.b In Eddington-Finkelstein coordinates the line element is

$$ds^2 = \left(1 - \frac{1}{r}\right) dv^2 - 2dvdr - r^2 d\Omega^2 \ , \tag{11}$$

[2]Ted's reference has a typo in equation (6).

[3]This quantity is really the "(energy) $-\ \Omega_H$(angular momentum)" flux, as it concerns the part of the energy-momentum tensor projected along the horizon-generating Killing field ξ.

where the advanced time coordinate is defined as $v = t + r_*$ in terms of the terrapin coordinate $r_* = r + \log(r-1)$. We want to know the form of the out-going null rays. We take, as usual, $d\Omega = 0$ and solve (11) for $ds^2 = 0$. Since we do not want the in-going null geodesic, $dv \neq 0$ and we find

$$dv = \frac{2dr}{1-\frac{1}{r}} = 2dr_* \ . \tag{12}$$

Integrating this equation between initial and final configurations with masses M and $M + \Delta M$ gives

$$\frac{\Delta r}{2M} + \log\left(\frac{r_f - 2M}{r_i - 2M}\right) = \frac{\Delta v}{2 \cdot 2M} \ . \tag{13}$$

Here we have restored the units which we had originally defined by setting $2M = 1$. Now, taking the initial radius $r_i = 2M(1+\epsilon)$ to be slightly larger than the Schwarzschild radius and the final radius $r_f = 2(M + \Delta M)$ to be the radius after the second shell collapses, $\Delta r = 2\Delta M$ and the equation becomes

$$\frac{\Delta M}{M} + \log\left(\frac{\Delta M}{M\epsilon}\right) = \kappa(v_f - v_i) \ . \tag{14}$$

Finally, discarding small terms[4] $O(\frac{\Delta M}{M})$ and exponentiating, we find that in the far past

$$\epsilon \sim a e^{\kappa v_i} \ , \tag{15}$$

where $a = (\Delta M/M)\exp(-\kappa v_f)$. We see that as $v_i \to -\infty$, $\epsilon \to 0+$ exponentially.

[4]It is not necessary to assume that these terms are small. Keeping them simply changes the coefficient a in (15).