Phys374, Spring 2008, Prof. Ted Jacobson
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Waves on a stretched string

A stretched string will vibrate when plucked. If the string is finite it oscillates, while if the string is infinite it supports traveling waves. The physical parameters defining the problem are the mass per unit length ρ and tension τ of the string.

Dimensional analysis

Exercise a: Show that the combination $\sqrt{\tau/\rho}$ has dimensions of velocity, *and* that one cannot make a dimensionless quantity using ρ, τ, and the wavelength λ. This allows us to infer that the wave speed is independent of wavelength and is proportional to $\sqrt{\tau/\rho}$.

Exercise b: If the string has fixed endpoints separated by a length ℓ then it can vibrate at a particular set of normal mode frequencies. The lowest frequency must be proportional to some combination of the available constants ρ, τ, and ℓ. Find this combination.

Wave equation

Here we use Newton's law to derive a partial differential equation describing the motion of the string. We suppose the equilibrium configuration of the string lies along the x axis, and we let $y(x,t)$ denote the perpendicular displacement of the string from its equilibrium at position x and time t. We assume that the displacement of the string is very small, in the sense that

$$\frac{\partial y}{\partial x} \ll 1 \tag{1}$$

which means that the *slope* of the string is everywhere very small compared to one. Equivalently the *angle* θ between the string and the horizontal is small.

Since different parts of the string have different motions, we need to apply Newton's law $\mathbf{F} = m\mathbf{a}$ to each infinitesimal bit of the string separately. To this end, consider the bit of string that runs from x to $x + dx$. Since we assume the slope is very small, the length of this bit of string is nearly just dx. The correction is of order $(dx)^2$. Neglecting this, as well as the related stretching of the string[1], we have that to lowest order in dx the mass of this bit of string is

$$m = \rho dx. \tag{2}$$

The acceleration of this bit in the y direction is, to zeroth order in dx, given by the second partial derivative with respect to t,

$$a_y = \frac{\partial^2 y}{\partial t^2} \tag{3}$$

evaluated at x.

It remains to evaluate the y component of the force on this bit of string. The force arises from the vector sum of the forces due to the pull of the string on the right and on the left of the bit.[2] The string exerts a force of magnitude equal to the tension τ, and direction along the string. The y component of the tension force on the left depends on the angle θ made by the tangent to the string at x and is

$$\begin{aligned} F_y^{\text{left}} &= -\tau \sin\theta && (4) \\ &= -\tau \tan\theta + O(\theta^3) && (5) \\ &= -\tau \frac{\partial y}{\partial x} + O\left[\left(\frac{\partial y}{\partial x}\right)^3\right]. && (6) \end{aligned}$$

The minus sign is because the string on the left pulls downward (in the negative y direction) if the slope is positive. Since we assume the slope is much smaller than 1, the correction term can be neglected in what follows, but it should be clear that our result for the force is only accurate up to a correction of relative size equal to the square of the slope. The string force

[1] The string mass is fixed. When the string is displaced from equilibrium and is therefore lengthened its mass per unit length must decrease. To take this into account we would have to allow for the string to have a nonuniform density, and also a nonuniform tension. This would require us to introduce another functional freedom, describing the deviation of the string from its equilibrium density at each x and t. The resulting system would be more complicated to handle. I read that for the bass strings on a piano this degree of freedom does play an important role in the nature and tone of the vibrations.

[2] We ignore any force due to resistance to *bending* the string since, under our assumption of small slope, it is reasonable to imagine that the bending force is negligible. Of course one can consider a very stiff "string", e.g. a fine metal rod, for which the bending force would be important or even dominant. But by "string" we mean something that has negligible resistance to bending the small amount considered here.

on the right is given by a similar expression, without the minus sign. Thus to lowest order in the slope the net force in the y direction due to tension is

$$F_y = \tau \frac{\partial y}{\partial x}\Big|_{x+dx} - \tau \frac{\partial y}{\partial x}\Big|_{x}. \tag{7}$$

This would be zero were it not for the slight difference in location where the slopes are evaluated. Since Newton's law equates this to ma_y with m of order dx, we only need to evaluate F_y to this order. To do so we expand the first term in dx keeping only the first order term. That is, we apply the relation $f(x+dx) = f(x) + f'(x)dx + O((dx)^2)$ to the first term. That is,

$$\frac{\partial y}{\partial x}\Big|_{x+dx} = \frac{\partial y}{\partial x}\Big|_{x} + \frac{\partial^2 y}{\partial x^2}\Big|_{x} dx + O((dx)^2), \tag{8}$$

so to $O(dx)$ F_y becomes

$$F_y = \tau \frac{\partial^2 y}{\partial x^2}\, dx. \tag{9}$$

Now we impose Newton's law $F_y = ma_y$, with (2), (3), and (9). Note that both F_y and m are of order dx, while a_y is of order 1, so the equation is sensible. Dividing by the common factor of dx we obtain the string equation of motion

$$\tau \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2}. \tag{10}$$

Equivalently, we can write

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0, \tag{11}$$

where $v = \sqrt{\tau/\rho}$. This is second order, linear partial differential equation is a wave equation for waves that travel at speed v.

General solution to wave equation

The wave equation (11) can be re-expressed in the form

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) y = 0. \tag{12}$$

The cross-terms cancel since partial derivatives commute, $\frac{\partial}{\partial t}\frac{\partial}{\partial x} y = \frac{\partial}{\partial x}\frac{\partial}{\partial t} y$.

The two derivative operator factors on the left hand side of (12) also commute (since v is a constant). Hence $y(x,t)$ is a solution to (12) if it satisfies

$$\left(\frac{\partial}{\partial t} \pm v\frac{\partial}{\partial x}\right) y = 0 \tag{13}$$

with either the plus sign or the minus sign. Solutions are given by

$$y(x,t) = f(x - vt) + g(x + vt) \tag{14}$$

where f and g are arbitrary functions of their arguments. The f solution represents a pattern moving in the positive x direction with velocity v, while the g solution pattern moves with velocity $-v$. Note that the unknown dimensionless coefficient in the wave speed turned out to be 1.

Exercise c: The above reasoning implies that (14) satisfies the wave equation (11). Verify this explicitly by direct substitution of (14) into (11).

It is not hard to show that (14) is in fact the *general* solution to the wave equation. Since that equation is second order, the initial data at time $t = 0$ (for example) that determine a solution are the two functions $y(x,0)$ and $\partial_t y(x,0)$. One can solve for f and g in terms of these two functions.

String with fixed endpoints

If the endpoints are fixed and separated by a length ℓ, the string motion is still governed by the wave equation, but at the endpoints we have the restriction $y(0,t) = y(\ell,t) = 0$. These boundary conditions can be satisfied by appropriate superpositions of right and left moving traveling waves that cancel out to zero at the two endpoints.

Exercise d: Among the solutions with fixed endpoints are the *normal modes*. These are solutions in which the time dependence is given by an overall sinusoidal factor with frequency ω:

$$y(x,t) = \sin(\omega t + \varphi) f(x) \tag{15}$$

(i) Insert this form into the wave equation (11) and determine the ordinary differential equation satisfied by $f(x)$. (ii) Find the general solution to this equation for $f(x)$. (iii) Impose the boundary conditions $y(0,t) = y(\ell,t) = 0$ to determine a discrete set of allowed frequencies ω, and the corresponding profiles $f(x)$. Any motion of the string is described by a sum of such normal mode solutions.