HW#9 —Phys374—Spring 2007
Due before class, Friday, April 13, 2007
www.physics.umd.edu/grt/taj/374b/

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1. Express the real and imaginary parts of the following functions in terms of $x = Re(z)$ and $y = Im(z)$: z^3, e^z, e^{iz}, $\sin z$, $1/(z^2+1)$.

2. Evaluate $\int (z^2 - z)dz$ along the following two contours connecting 0 to $1+i$: (a) from 0 to 1 along the real axis and then 1 to $1+i$ along the imaginary direction; (b) along the diagonal directly from 0 to $1+i$. (c) Verify that you obtain the same result either way, and explain how you could have known the two integrals would be the same without even evaluating them.

3. Find the residues of the following functions at the given values of z:

   (a) $(z+z^2)^{-1}$ at 0 and at -1.
   (b) $\ln(1+2z)/z^2$ at 0
   (c) $[z^3(z+2)^2]^{-1}$ at 0 and at -2.
   (d) $\cos z/(2z-\pi)^4$ at $\pi/2$
   (e) $(z^2+1)^{-3}$ at $\pm i$.

   Hint: See the supplement for methods of evaluating residues.

4. Consider the real integral
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$
   where a and b are positive real numbers.

   (a) Evaluate the integral using contour integration assuming $a \neq b$ (so there are only simple poles).
   (b) Evaluate the integral using contour integration assuming $a = b$ from the beginning (so the poles are of order 2), and then check that you recover the same result by setting $a = b$ in the result of the previous part.

   Hint: You can check your result by testing for a few properties: from the form of the integral it is manifestly positive and symmetric under interchange of a and b, and if you think of a and b as having dimensions, the integral has dimensions of $[a]^{-3}$.

5. (a) Show using contour integration that
$$\int_0^{\infty} \frac{\cos mx\, dx}{x^2+a^2} = \frac{\pi}{2a}\epsilon^{-ma}$$
   Hint: See the similar example in the textbook.

   (b) Explain in words why the result decays so rapidly (i) as m grows with a fixed, and (ii) as a grows with m fixed.

6. Evaluate the integral $\int_0^{\infty} dx/(x^3+1)$ by relating it to a contour integral over the boundary of an infinite piece of pie with edges $\theta = 0$ and $\theta = 2\pi/3$, together with the arc at infinity that joins these edges. The answer is $2\pi/3^{3/2}$.

7. Problem 16.1h

8. Problem 16.3 c,d,e,f

9. Consider the flow of a fluid that is incompressible and has no vorticity, so this is "potential flow". Assume the flow velocity vector is constant and points in the x direction, $\mathbf{v} = v_0\hat{\mathbf{x}}$. The velocity potential f satisfies Laplace's equation, and we can think of this as a two-dimensional problem in the xy plane.

   (a) Find an analytic function $h_1(z)$ of $z = x + iy$ whose real part is the velocity potential for this flow.

   (b) What are the flow lines for this flow?

   (c) How exactly are the flow lines related to $h_1(z)$?

10. Consider potential flow in the xy plane past an infinite cylinder of radius R perpendicular to that plane. [*This problem is worth 20 pts.*]

    For the boundary condition "at infinity", suppose that far from the cylinder in all directions the velocity is as in the previous problem, $\mathbf{v} = v_0\hat{\mathbf{x}}$. The boundary condition at the cylinder surface is that there is no flow perpendicular to the cylinder, so $\mathbf{v}\cdot\hat{\mathbf{r}} = 0$, taking the origin of polar coordinates at the center of the cylinder. That is, the partial derivative of the velocity potential with respect to radius r (at fixed angle) vanishes at $r = R$.

    To solve for the flow we need only find an analytic function of $z = x + iy$ whose real part satisfies the appropriate boundary conditions.

    (a) Find an analytic function $h_2(z)$, to be added to your function $h_1(z)$ from the previous problem, such that the real part of $h(z) = h_1(z) + h_2(z)$ satisfies the boundary condition everywhere on the cylinder, as well as the boundary condition at infinity. To do this assume $h_2(z) = az^n$, and find values for constants a and n so that the boundary conditions are satisfied. (*Hint*: Use polar coordinates.)

    (b) Using your result from the previous part, find the velocity (magnitude and direction) at the point $(x, y) = (0, R)$ on the surface of the cylinder.

    (c) Explain why the flow lines are given by the contours of the imaginary part of $h(z)$.

    (d) Find the equation for the flow line that goes through the point $(x, y) = (0, y_0)$ (with $y_0 > R$). (The equation should involve x, y, y_0, v_0 and R.)

    (e) Find the asymptotic y value y_∞ when $x \to \infty$ for the flow line of problem 10d.

    (f) Sketch the flow line of problem 10d along with the circle of radius R representing the cross-section of the cylinder.