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Department of Physics, University of Maryland

Gravity waves on water

Waves on the surface of water can arise from the restoring force of gravity or of surface tension, or a combination. For wavelengths longer than a couple of centimeters surface tension can be neglected, and the waves are called *gravity waves*. Short wavelength surface waves are called *capillary waves*. Dimensional analysis told us that the speed of gravity waves with wavelength λ much shorter than the depth of the body of water but still long enough to ignore surface tension must be proportional to $\sqrt{g\lambda}$, while those with wavelength much longer than the depth have speed proportional to $\sqrt{gh}$. In this supplement we'll derive the speed for the general case.

Fluid equation of motion

The starting point is to apply $\mathbf{F} = m\mathbf{a}$ to the fluid, or rather to each infinitesimal fluid element. The mass of such an element per unit volume is ρ, the mass density. The acceleration of a fluid element is the rate of change of its velocity $\mathbf{v}(\mathbf{r}, t)$ as it is carried along by the flow. This receives contributions from any explicit time dependence as well as from the fact that the flow may carry it into a location where the velocity field is different. That is, the acceleration is the *convective derivative* of the velocity,

$$\mathbf{a} = \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{1}$$

(See section 5.5 of the textbook for a discussion of the convective derivative.) The force on a fluid element per unit volume is the pressure force $-\nabla p$ plus the gravitational force $-\rho\nabla\Phi$, where Φ is the gravitational potential. Viscosity would produce another force, but it is negligible for this problem. The acceleration of a fluid element is equal to the force per unit volume divided by the mass per unit volume, hence

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\rho^{-1}\nabla p - \nabla\Phi. \tag{2}$$

This is our fluid equation of motion. It's called the *Euler equation*, and is the *Navier-Stokes equation* with viscosity terms neglected.

Incompressible flow

Mass conservation is expressed by the continuity equation, $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$. If, like water (but unlike, say, a gas), the fluid is very difficult to compress, then the mass density ρ will be nearly constant in space and time, so the continuity equation reduces to $\nabla \cdot \mathbf{v} = 0$. This is simply the statement that the net volume flow into or out of any region is zero.

Irrotational flow

The vorticity of the flow is defined as $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. A remarkable fact is that, for incompressible flow, only viscosity can generate vorticity if none exists initially. To see why, just take the curl of (2). The right hand side is a gradient if ρ is constant, so its curl vanishes. The curl of the first term on the left hand side is just $\partial_t \boldsymbol{\omega}$. The second term is the only tricky one. For that we need an identity:

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} + \nabla(v^2/2). \tag{3}$$

This is easy to prove using an index notation. The second term is a gradient so has vanishing curl. Thus we've found the evolution law for the vorticity,

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = 0. \tag{4}$$

If the vorticity is initially zero, then according to this equation its time derivative is zero, so it remains zero. It is thus consistent to assume the vorticity is zero in the wave, i.e. the flow is *irrotational*.

Since the velocity has zero curl for irrotational flow it must be the gradient of a scalar, $\mathbf{v} = \nabla f$. The function f is called the *velocity potential*, and the flow is also called *potential flow*. If an irrotational flow is also incompressible, then $\nabla^2 f = 0$, i.e. its velocity potential satisfies Laplace's equation.

Exercise a: Show that the vorticity equation (4) holds even if the flow is not incompressible, provided the density is a function of the pressure, i.e. it holds for a *barotropic* fluid.

Fixing a wavelength

The first step in deriving the form of a gravity wave with wavelength λ is to assume the velocity potential $f(\mathbf{r}, t)$ has a factorized form with this wavelength,

$$f(\mathbf{r},t) = \mathrm{Re}\Big(A(t)G(y)e^{ikx}\Big) \tag{5}$$

where the wave number is $k = 2\pi/\lambda$ and $\mathrm{Re}(z)$ means the real part of a complex number z. Here we have assumed that $\hat{\mathbf{x}}$ is the wave propagation direction, $\hat{\mathbf{y}}$ is the vertical direction, and the wave is completely independent of the remaining direction $\hat{\mathbf{z}}$. This means the wave crests will be infinitely long straight lines in the z-direction. We choose the surface of the undisturbed fluid to lie at $y = 0$. The general solution will be a linear combination of solutions of this factorized form, though we won't prove that here.

Imposing the incompressibility condition

Next we impose the condition that f satisfies Laplace's equation. Since we fixed the wavelength and assumed no z-dependence, this yields

$$\begin{aligned} \nabla^2 f &= (\partial_x^2 + \partial_y^2 + \partial_z^2) f & (6) \\ &= (-k^2 + G''/G) f & (7) \end{aligned}$$

where G'' stands for d^2G/dy^2. Laplace's equation $\nabla^2 f = 0$ thus reduces to a simple ordinary differential equation for $G(y)$,

$$G'' = k^2 G. \tag{8}$$

The general solution to this equation is

$$G(y) = c_1 e^{ky} + c_2 e^{-ky}. \tag{9}$$

Boundary condition at the bottom

To select the solution relevant to our problem we must think about what is happening at the boundaries. First we consider the bottom, then the free surface at the top.

If the container is infinitely deep the boundary condition is simplest: $G(y \to -\infty) = 0$. Using (9) this implies $c_2 = 0$. We should not try to determine the other constant c_1, since it represents just the overall scale of the wave amplitude, which is free.

If the container has a finite depth h, then the boundary condition to be imposed at $y = -h$ is that the vertical component of velocity vanishes there (no flow through the bottom). To impose this, remember that $\mathbf{v} = \nabla f$, so

$v_y = 0 \iff \partial_y f = 0 \iff G' = 0$. Thus our boundary condition at the bottom is $G'(-h) = 0$. Since

$$G'(y) = c_1 k e^{ky} - c_2 k e^{-ky} \tag{10}$$

we infer $c_2 = e^{-2kh} c_1$. When $h \to \infty$ this yields $c_2 = 0$, consistent with what we got previously. For a general h we have

$$\begin{aligned} G(y) &= c_1(e^{ky} + e^{-2kh} e^{-ky}) && (11) \\ &= c_1 e^{-kh}(e^{k(y+h)} + e^{-k(y+h)}) && (12) \\ &= 2c_1 e^{-kh} \cosh(k(y+h)). && (13) \end{aligned}$$

At the bottom ($y = -h$) the cosh equals 1, while at the top it's $\cosh kh$, so G grows as y goes from bottom to top values. The constant coefficient $2c_1 e^{-kh}$ can be absorbed into the definition of $A(t)$ in (5).

So far we have only imposed the conditions that the flow is irrotational and incompressible. These don't capture the full content of the fluid equation of motion (2) which, remember, is just expressing Newton's law for every fluid element. The time has come to impose Newton's law.

Small amplitude assumption

For the incompressible, irrotational flow, every term in (2) is a gradient except for $(\mathbf{v} \cdot \nabla)\mathbf{v}$, which is called the *advective* term in the equation. (See section 12.3 of the textbook for a discussion of this.) We are now going to make a further physical assumption about the nature of the waves that renders this term negligible.

The advective term is quadratic in $\mathbf{v}$, whereas the first term in (2) is linear and the force terms are independent of $\mathbf{v}$. For small velocities you might therefore expect that the advective term is negligible, but what exactly does "small" mean? A velocity has dimensions LT^{-1}, so it is meaningless to say it is "small" in absolute terms. Only dimensionless numbers can be small in absolute terms. What we really must mean is that it is small compared to something else that has the *same* dimensions. If the issue is whether or not we can drop the term in the equation, then the question is whether it is small compared to the other terms in the equation.

Thus what we really want to do is to compare the size of $(\mathbf{v} \cdot \nabla)\mathbf{v}$ with $\partial_t \mathbf{v}$. Referring to the form of f that we've found so far, you can see that each spatial derivative, either with respect to x or y, brings in a factor of $k = 2\pi/\lambda$. Thus spatial derivatives contribute $1/\lambda$ factors. Similarly, the time derivative will contribute a $1/T$ factor, where T is the period of the oscillation. Also the velocity will have an overall amplitude v_0. In this way we make the following size estimates:

$$(\mathbf{v}\cdot\nabla)\mathbf{v} \sim \frac{v_0^2}{\lambda} \qquad \partial_t \mathbf{v} \sim \frac{v_0}{T}. \tag{14}$$

Thus we've found

$$\frac{\text{advective term}}{\text{time derivative}} \sim \frac{v_0 T}{\lambda} = \frac{v_0}{c}, \tag{15}$$

where $c = \lambda/T$ is the wave velocity. From this we conclude that the advective term is negligible provided the fluid velocity amplitude at the top is much smaller than the wave speed. Since we don't yet know the wave speed this is not easy to assess, so lets back up one step in (15). The numerator $v_0 T$ is the maximum distance a fluid element travels in one period of the wave. That is, it's the amplitude of the wave motion at the surface. If this is much smaller than the wavelength, then the advective term can be neglected. Let's make this assumption from now on. This assumption *linearizes* the equation of motion and makes the problem *much* easier to solve.

Boundary condition at the top

What can be the boundary condition at the top? What is fixed there? The surface is moving up and down, but the pressure is always the same and equal to the atmospheric pressure p_{atm} above the surface. This will be our boundary condition, but how can it be imposed on the solution? The constancy of the pressure says something about the force, so to impose it we must go back to the the fluid equation of motion (2).

For waves whose amplitude of motion is smaller than the wavelength, all significant terms in the fluid equation (2) are gradients, and the equation can be expressed as

$$\nabla(\partial_t f + p/\rho + \Phi) = 0. \tag{16}$$

Hence under these assumptions the quantity $\partial_t f + p/\rho + \Phi$ is constant in space. Since $p = p_{\text{atm}}$ everywhere at the surface we infer that *on the surface*

$$(\partial_t f + \Phi)\Big|_{y=y_s} = F(t), \tag{17}$$

where $y = y_s(x,t)$ is the y coordinate of the surface at position x and time t, and $F(t)$ is some undetermined function depending on time but not on space. We can set $F(t) = 0$ without loss of generality, since it could always be absorbed into $f(\mathbf{r},t)$ by the spatially constant shift $f \to f - \int^t F(t)\,dt$ which has no physical effect since the velocity is $\mathbf{v} = \nabla f$.

Now the gravitational potential is $\Phi = gy$ (up to an arbitrary constant). To relate the vertical position of the surface y_s to the velocity potential f we take a time derivative of (17) (with $F(t) = 0$), yielding $\partial_t^2 f + g\partial_t y_s = 0$ at the surface. Referring to the factorized form of f (5) we have $\partial_t^2 f = (\ddot{A}/A)f$, where $\ddot{A}$ stands for d^2A/dt^2. Moreover

$$\begin{aligned} \partial_t y_s &= \partial_y f && (18) \\ &= (G'/G)f && (19) \\ &= k \tanh(k(y_s + h))f, && (20) \end{aligned}$$

all terms being evaluated at $y = y_s$. This yields for $A(t)$ the equation

$$\ddot{A}/A = -gk \tanh(k(y_s + h)). \quad (21)$$

Smaller amplitude assumption

Now it seems we are stuck! The left hand side is only a function of t, while the right hand side is a function of $y_s(x,t)$. This means that there is no solution!! However, remember we are making a small amplitude approximation. We have already neglected terms of order y_s/λ. If the depth is greater than the wavelength, then this means we should also neglect the ratio y_s/h, and therefore we can replace $\tanh(k(y_s+h))$ by the constant expression $\tanh(kh)$. But how about if the depth is *not* greater than the wavelength? Then we are sunk, unless we make the *additional* assumption that the amplitude is much smaller than the depth.

Dispersion relation for gravity waves

With the assumption $y_s \ll h$ we can safely neglect the y_s in (21), which then becomes just the harmonic oscillator equation,

$$\ddot{A} = -[gk \tanh kh]A. \quad (22)$$

The general complex solution is given by $A(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$, where $A_{1,2}$ are constant amplitudes and

$$\omega = \sqrt{gk \tanh kh}. \quad (23)$$

This is the *dispersion relation* between frequency ω and wavevector k for incompressible, irrotational, "small" amplitude gravity waves on a fluid surface. (See homework 2, problem 3.) Putting together what we've found, the velocity potential (5) is given for the right-moving solution (the one with $A_1 = 0$, assuming $k > 0$) by

$$f(x, y, z, t) = \mathrm{Re}\Big(A_0 e^{i(kx-\omega t)} \cosh(k(y+h))\Big) \tag{24}$$

In the case $kh \gg 1$ the cosh shuld be replaced by $\exp(ky)$.

Motion of the fluid

Consider the trajectory of a fluid element, given by the position vector $\boldsymbol{\xi}(t)$. The equation defining the trajectory is

$$\frac{d\boldsymbol{\xi}}{dt} = \mathbf{v}(\boldsymbol{\xi}, t). \tag{25}$$

To determine one such trajectory we can start somewhere in the fluid at some time and, given the function $\mathbf{v}(\mathbf{r}, t)$, integrate the flow equation (25). This is complicated however, since we have to keep track of where the fluid element flowed to at time t and evaluate $\mathbf{v}$ at that new location and time in order to determine where the fluid element goes from there.

If the fluid elements don't go very far from their equilibrium locations, however, a great simplification ensues: to find the motion near some starting point $\mathbf{r}_0$ it suffices to a good approximation to just evaluate $\mathbf{v}$ at $\mathbf{r}_0$. Then finding the trajectory just requires integrating with respect to time with $\mathbf{r} = \mathbf{r}_0$ held fixed. More explicitly, let the position of the fluid element be given by $\boldsymbol{\xi}(t) = \mathbf{r}_0 + \boldsymbol{\eta}(t)$. Then the trajectory equation (25) becomes

$$\frac{d\boldsymbol{\eta}}{dt} = \mathbf{v}(\mathbf{r}_0 + \boldsymbol{\eta}, t) \approx \mathbf{v}(\mathbf{r}_0, t). \tag{26}$$

The approximation is good when $(\boldsymbol{\eta} \cdot \nabla)\mathbf{v} \ll \mathbf{v}$. Remember now our scale analysis above for the advective term. The spatial derivative scales like the inverse wavelength $1/\lambda$. On the other hand $\boldsymbol{\eta}$ is the fluid displacement. So if the fluid displacements are small compared to the wavelength, the approximation will be good. Once this approximation is made, the problem is easy: (26) then implies that the motions around $\mathbf{r}_0$ are given by

$$\boldsymbol{\xi}(t) = \mathbf{r}_0 + \int^t \mathbf{v}(\mathbf{r}_0, t)\, dt. \tag{27}$$

The lower limit is unspecified since there is an initial displacement from $\mathbf{r}_0$ that is arbitrary as long as it is small.

Exercise b: (i) Show that the trajectory of a fluid element near a point in the fluid with coordinates (x_0, y_0) is an elliptical motion with the same period as the wave and a size that is smaller for deeper y_0. (For shallow waves the amplitude of the horizontal motion is fairly constant with depth.) (ii) Show that if the water is deep, $kh \gg 1$, the ellipses are nearly circles at the surface and remain so until y_0 is very close to $-h$. (iii) If the wave moves to the right, is the elliptical motion clockwise or counterclockwise? (Justify your answer with reference to your equations.)