(2·1 solution is at end)

HW2 | solutions ①

$\boxed{2\cdot 2\cdot 1}$ Calculation of $F_1(0)$

First, calculate UV-finite contribution from 3rd term in Eq. (11·81) of Lahiri & Pal contains

$$\int \overset{\leftarrow \text{ no need to use DIMREG}}{\frac{d^4k}{(2\pi)^4}} \int dx_{1,2} \frac{-2m^2\left[(x_1+x_2)^2 - 2(1-x_1-x_2)\right]}{\left[k^2-(x_1+x_2)^2 m^2\right]^3} \quad \ldots (1)$$

$$= \int_0^{1 \text{ or } 1-x_1} dx_{1,2} \frac{-2\cancel{m^2}\, i(-1)^3}{(4\pi)^2\left[(x_1+x_2)^2\cancel{m^2}\right]} \times \left\{(x_1+x_2)^2 - 2\left[1-(x_1+x_2)\right]\right\} \leftarrow \text{use } r=0, s=3 \text{ in Eq. (A·54)}$$

$$\times \frac{\Gamma(2)\Gamma(1)}{\Gamma(3)} \Big\}1 \leftarrow 2$$

Feynman parameter integral : $\int_1^1 dx_1 \int_0^{1-x_1} dx_2$ ← due to $\theta(1-x_1-x_2)$ in Eq. (11·80)

$$\times \left[1 - \frac{2}{(x_1+x_2)^2} + \frac{2}{(x_1+x_2)}\right]$$

$$= \left[\int_0^1 dx_1 (1-x_1)\right] + \left[2\int_0^1 dx_1 \left(\frac{1}{x_1+x_2}\right)\Bigg|_{x_2=0}^{(1-x_1)}\right] + \left[2\int_0^1 dx_1 \log(x_1+x_2)\Bigg|_0^{1-x_1}\right]$$

$$= \left[\frac{1}{2}\right] + \left[2\int_0^1 dx_1\left(1-\frac{1}{x_1}\right)\right] + \left[2\int_0^1 dx_1 (0-\log x_1)\right]$$

$$= \left[\tfrac{1}{2}\right] + \left[2 - 2\log x_1\Big|_0^1\right] + \left[-2 \quad x_1(\log x_1 - 1)\Big|_0^1\right]$$

$$= \left[\tfrac{1}{2}\right] + \left[\begin{matrix}\text{divergent} \\ \text{due to } \log(0)\end{matrix}\right] + 2 \quad \left(\lim_{x_1\to 0} x_1 \log x_1 = 0\right) \ldots (2)$$

⇒ need to regulate (only) middle Feynman parameter integral by non-zero photon mass :

$x_3 k^2$ in D of Eq. (11.43) goes to $x_3(k^2 - m_\gamma^2)$ (since this term is from photon propagator: see Fig. 11.1) ...

Also, it's more convenient to use $\delta(1-x_1-x_2-x_3)$ in Eq. (11.46) to rewrite (x_1+x_2) in denominator of Eqs. (11.47) and (11.80) (recall $q^2=0$ there) in terms of x_3, i.e., the (IR-)divergent term in Eq. (1) above becomes

$$\int_0^1 dx_3 \int \frac{d^4k}{(2\pi)^4}\, \frac{-2m^2(-2)}{\Big[k^2 - \underbrace{(1-x_3)^2}_{\text{was } (x_1+x_2)^2 \text{ earlier}} \underset{\text{electron mass}}{m^2} - x_3 \underset{\text{IR-regulator}}{m_\gamma^2}\Big]^3}$$

$$\times \int_0^1 dx_{1,2}\, \delta(1-x_1-x_2-x_3)$$

$$= \int dx_3\, (+4m^2)\, \frac{i(-1)^3}{(4\pi)^2}\, \frac{1}{\left[(1-x_3)^2 m^2 + x_3 m_\gamma^2\right]} \times \boxed{\frac{1}{2}} \leftarrow \text{from } \Gamma\text{'s} \ldots$$

$$\times \left.\int_0^1 dx_1\, \theta(1-x_1-x_3)\right\}\text{gives} \int_0^{1-x_3} dx_1 = (1-x_3)$$

Feynman parameter integral which replaces 2nd (divergent) term in Eq. (2) above is then

$$(-2)\int dx_3\, \underbrace{(1-x_3)}_{\parallel} \Big/ \left[(1-x_3)^2 + x_3 m_\gamma^2/m^2\right] = \log\left[(1-x_3)^2 + x_3 \frac{m_\gamma^2}{m^2}\right]\Bigg|_0^{x_3=1}$$

$$\frac{1}{2}\frac{m_\gamma^2}{m^2} - \left(\frac{1}{2}\right)\left[-2(1-x_3) + m_\gamma^2/m^2\right]\Big) - \frac{m_\gamma^2}{m^2}\int dx_3 \Big/ \left\{\left[x_3 - \left(1 - \tfrac{1}{2} m_\gamma^2/m^2\right)\right]^2 + m_\gamma^2/m^2 - \tfrac{1}{4} m_\gamma^4/m^4\right\}$$

$$\simeq \log\left(m_\gamma^2/m^2\right) + \underbrace{O\left(\frac{m_\gamma^2}{m^2}\right)\frac{1}{\left(\frac{m_\gamma}{m}\right)}\underbrace{\tan^{-1}\left[x_3-\left(1-\frac{1}{2}\frac{m_\gamma^2}{m^2}\right)\right]}_{O\left(m_\gamma/m\right)}}_{\text{drop since } \rightarrow 0 \text{ as } \frac{m_\gamma}{m}\rightarrow 0}$$

$\Rightarrow$ UV- <u>finite</u> part of $F_1(0)$

$$= \underbrace{\frac{8\pi i\alpha}{\Gamma(3)\leftarrow 2}\;\frac{+2i}{16\pi^2}}_{-\alpha/(2\pi)} \times \underbrace{\left(\frac{1}{2}+\log\frac{m_\gamma^2}{m^2}+2\right)}_{\text{modified Eq. (2) above}}$$

[Note that had we set $m_\gamma = 0$ in above $\int dx_3$, we would get $2\int_0^1 dx_3\,(1-x_3)/(1-x_3)^2$

$= -2\int_0^1 dx/x = \left[-2\log x\Big|_0^1\right]$ ... instead

of $\left[2-2\log x\Big|_0^1\right]$ in 2nd term of Eq. (2)

[where x_3 was gotten rid of using $\delta(1-x_1-x_2-x_3)$

instead of...], i.e., there's "extra" 2 in earlier form: this just underscores the point that order of integration does matter when there's divergence (or we need to regulate if we wish to compare F_1 to Z_2 ...)

← no need to use $m_\gamma \neq 0$

Next, calculate UV - divergent (IR finite) part of $F_1(0)$ from 1st term in Eq. (11.50) of Lahiri & Pal :

(i) DIMREG

$$\Pi_\mu^{1\text{-loop}}(q^2=0) = 8\pi i\alpha \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \int_0^1 dx_1 \int_0^1 dx_2 \times$$

$$\theta(1-x_1-x_2)\,\frac{\gamma_\lambda \not{k}\, \gamma_\mu \not{k}\, \gamma^\lambda}{\left[k^2-(x_1+x_2)^2 m^2\right]^3} \times \mu^{2\varepsilon} \quad \ldots(1)$$

$$\text{Use}\quad \gamma_\lambda \overset{\times k^\nu}{\overset{\downarrow}{\gamma_\nu}} \gamma_\mu \overset{k^\rho}{\overset{\downarrow}{\gamma_\rho}} \gamma^\lambda = -2\,\gamma_\rho\gamma_\mu\gamma_\nu + 2\varepsilon\,\gamma_\nu\gamma_\mu\gamma_\rho$$

and $\int (d^{4-2\varepsilon}k) k^\nu k^\rho f(k^2) = \int d^{4-2\varepsilon}k \,\frac{k^2}{4-2\varepsilon}\, f(k^2) g^{\nu\rho}$

to simplify γ - matrix part of numerator in Eq. (1) above to $\frac{g^{\nu\rho}k^2}{(4-2\varepsilon)}\left(-2\gamma_\rho\gamma_\mu\gamma_\nu + 2\varepsilon\gamma_\nu\gamma_\mu\gamma_\rho\right)$

$$= \frac{k^2}{(4-2\varepsilon)}\left(-2\gamma_\rho\gamma_\mu\gamma^\rho + 2\varepsilon\,\gamma_\rho\gamma_\mu\gamma^\rho\right)$$

Then use $\gamma_\rho\gamma_\mu\gamma^\rho = -2(1-\varepsilon)\gamma_\mu$ to get $\frac{k^2}{(4-2\varepsilon)}\left[-2(1-\varepsilon)\right]^2\gamma_\mu$ and Eq. (A.53) ~~to~~ do $\int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}}$ (with $r=1,\ s=3$) :

$$\frac{i(-1)^4}{(4\pi)^{2-\varepsilon}}\;\frac{\Gamma(3-\varepsilon)\,\Gamma(\varepsilon)}{\Gamma(2-\varepsilon)\,\overset{=2}{\Gamma(3)}}\;\frac{1}{\left[(x_1+x_2)^2 m^2\right]^{\varepsilon}}\;\frac{4\,(1-\varepsilon)^2}{(4-2\varepsilon)}$$

Note that $\Gamma(\varepsilon) \simeq 1/\varepsilon - \gamma_E$ so that we need to keep $O(\varepsilon)$ terms in <u>other</u> factors: they give $O(1)$ terms upon multiplying by $1/\varepsilon$ from $\Gamma(\varepsilon)$:

i.e., $\mu^{2\varepsilon} \simeq 1 + \varepsilon \log \mu^2$; $\underline{\dfrac{\Gamma(3-\varepsilon)}{\Gamma(2-\varepsilon)} = \dfrac{\cancel{\Gamma(2-\varepsilon)}(2-\varepsilon)}{\cancel{\Gamma(2-\varepsilon)}} \simeq 2(1-\varepsilon/2)}$;

$$\frac{1}{(4\pi)^{2-\varepsilon}} \simeq \frac{1}{16\pi^2}\left(1 + \varepsilon \log 4\pi\right); \quad \frac{1}{\left[(x_1+x_2)^2 m^2\right]^{\varepsilon}} ; \simeq 1 - \varepsilon \log\left[(x_1+x_2)^2 m^2\right]$$

$$\frac{1}{4-2\varepsilon} \approx \frac{1}{4}\left(1 + \varepsilon/2\right) \text{ and } (1-\varepsilon)^2 \simeq 1 - 2\varepsilon$$

so that, combining above terms, we finally have

$F_1(0)$ from UV-divergent part of loop

$$= 8\pi i \alpha \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{i}{16\pi^2}\left\{\frac{1}{\varepsilon} - \gamma_E + \left(\frac{1}{\varepsilon}\right)\cdot \varepsilon \left(\log \mu^2 - 1/2 + \log 4\pi - \log\left[(x_1+x_2)^2 m^2\right] + 1/2 - 2\right)\right\}$$

from $\Gamma(\varepsilon)$ ↗ $\left(\frac{1}{\varepsilon}\right)$; ↑ from other factors above

$$= -\frac{\alpha}{2\pi}\left\{\int_0^1 dx_1 (1-x_1)\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log(\mu^2/m^2) - 2\right] - 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \log(x_1+x_2)\right\}$$

$$= -\frac{\alpha}{2\pi}\left\{\underbrace{\left(x_1 - \frac{x_1^2}{2}\right)\Big|_0^1}_{=1/2}\left[1/\varepsilon - \gamma_E + \log 4\pi + \log(\mu^2/m^2) - 2\right] + \right.$$

(continued on next page)

$$-2\int dx_1\,(x_1+x_2)\left[\log(x_1+x_2)-1\right]\Big|_{0}^{x_2=1-x_1}\Big\}$$

The remaining $\int dx_1 = \int dx_1\left(-1-x_1\log x_1+x_1\right)$

$$=\left(-x_1+\frac{x_1^2}{2}\right)\Big|_0^1-\left[\frac{x_1^2}{2}\log x_1\Big|_0^1-\int_0^1 dx_1\,\frac{x_1^2}{2}\,\frac{1}{x_1}\right]$$

$$=-\tfrac{1}{2}-\left[0-\frac{1}{4}\right]=-\tfrac{1}{4}$$

$\Rightarrow$ $F_1(0)$ from UV-divergent part of loop

$$=-\frac{\alpha}{2\pi}\left\{\frac{1}{2\varepsilon}-\frac{\gamma_E}{2}+\frac{1}{2}\log 4\pi+\frac{1}{2}\log \mu^2/m^2-\tfrac{1}{2}\right\}$$

<u>Combining</u> UV finite & divergent parts of loop gives (in <u>DIMREG</u>)

$$F_1(0)=-\frac{\alpha}{2\pi}\left\{\frac{1}{2}\left(\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\log \mu^2/m^2\right)+2\;{}^{+\log m_\gamma^2/m^2}_{\wedge}\right\}$$

$\boxed{2.2.2}$ $\boxed{(i)}$ $Z_2-1=\ominus\, a_{CT}$ in $\boxed{\text{DIMREG}}$

$$=a(m^2)+2m^2\frac{da}{dp^2}\Big|_{p^2=m^2}+2m\frac{db}{dp^2}\Big|_{p^2=m^2}$$

with $a(p^2)$, $b(p^2)$ calculated in ${}^{\text{HW}}_{\wedge}$ 1.2.2 above $\Rightarrow$

$$a(m^2)=\frac{\alpha}{2\pi}\left[-\frac{1}{2\varepsilon'}+\frac{1}{2}+\int dx\,(1-x)\log\left(\frac{x^2m^2}{\mu^2}\right)\right]$$

$$= \frac{\alpha}{2\pi}\left[-\frac{1}{2\varepsilon'} + \frac{1}{2} + \left(x - \frac{x^2}{2}\right)\Big|_0^1 \log\frac{m^2}{\mu^2} + 2\; x(\log x - 1)\Big|_0^1\right.$$

$$\left. - 2\left(x^2/_2 \log x\,\Big|_0^1 - \int_0^1 dx\,\frac{x^2}{2}\,\frac{1}{x}\right)\right]$$

$$= \frac{\alpha}{2\pi}\left[-\frac{1}{2\varepsilon'} + \frac{1}{2} + \frac{1}{2}\log m^2/\mu^2 - 2 + 2\left(\frac{x^2}{4}\right)\Big|_0^1\right] \quad \cdots (1)$$

$$= \frac{\alpha}{2\pi}\left[-\frac{1}{2\varepsilon} + \frac{\gamma_E}{2} - \frac{1}{2}\log 4\pi + \frac{1}{2}\log\frac{m^2}{\mu^2} - 1\right],$$ i.e., IR-finite & UV-divergent

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$$\boxed{2m^2\frac{da}{dp^2}\Big|_{p^2=m^2} + 2m\frac{db}{dp^2}\Big|_{p^2=m^2}} = 2m^2\frac{\alpha}{2\pi}\int_0^1 \frac{dx\,(1-x)(-x)(1-x)}{[xm^2 - x(1-x)\,m^2]} - 2\frac{\alpha m^2}{\pi}\int\frac{dx\,(-x)(1-x)}{[xm^2 - x(1-x)\,m^2]}$$

$$= \frac{2\alpha m^2}{2\pi}\int_0^1 \frac{dx\,(-x)}{x^2m^2}(1-x)\left(1-x-2\right) \quad \cdots (2)$$

$$= \frac{\alpha}{\pi}\int_0^1 dx\; x\,(1-x^2)/x^2 = \frac{\alpha}{\pi}\left[\int_0^1\frac{dx}{x} - x^2/_2\Big|_0^1\right]$$

$$= \frac{\alpha}{\pi}\left[\log x\,\Big|_0^1 - 1/_2\right]$$

divergent due to log(0) : regulate using non-zero photon mass : x^2 in denominator of integrand in Eq. 2 comes from $xm^2 - x(1-x)\,p^2 (= m^2)$ which is modified to $xm^2 - x(1-x)\,p^2 + (1-x)\,m_\gamma^2$ since $(1-x)\,k^2$ from photon propagator in 1st line of Eq. (12.22) of Lahiri & Pal becomes $(1-x)(k^2 - m_\gamma^2)$

Thus $\int_0^1 dx\, x/x^2$ (divergent part above) becomes

$$\int_0^1 dx\; x\Big/\left[x^2+(1-x)m_\gamma^2/m^2\right] = \int_0^1 dx\, \frac{\left[\frac{1}{2}\left(2x-m_\gamma^2/m^2\right)+\frac{1}{2}\frac{m_\gamma^2}{m^2}\right]}{\left[x^2+(1-x)m_\gamma^2/m^2\right]}$$

$$=\frac{1}{2}\log\left[x^2+(1-x)m_\gamma^2/m^2\right]\Big|_0^1+\frac{1}{2}\frac{m_\gamma^2}{m^2}\times\underbrace{\int_0^1\frac{dx}{\left[\left(x-\frac{1}{2}\frac{m_\gamma^2}{m^2}\right)^2+\frac{m_\gamma^2}{m^2}-\frac{m_\gamma^2}{4m^2}\right]}}_{O\left(\frac{m_\gamma^2}{m^2}\right)\frac{1}{\frac{m_\gamma}{m}\left[1+O\left(\frac{m_\gamma^2}{m^2}\right)\right]}\tan^{-1}\left[\frac{x-\frac{1}{2}m_\gamma^2/m^2}{\frac{m_\gamma}{m}(1+\cdots)}\right]}$$

$$=-\frac{1}{2}\log\frac{m_\gamma^2}{m^2}+\underset{\text{as } m_\gamma\to 0}{\text{vanishing}}$$

Combining Eqs (1) & (2) above gives

$$\boxed{(Z_2-1)} = \frac{\alpha}{2\pi}\left[-\frac{1}{2\varepsilon}+\frac{\gamma_E}{2}-\frac{1}{2}\log 4\pi+\frac{1}{2}\log\frac{m^2}{\mu^2}-\log m_\gamma^2/m^2-2\right]$$

in $\boxed{\text{DIMREG}}$

$\Rightarrow$

$\boxed{2.2.3}$ $\boxed{(i)}$ $Z_1 = Z_2$ in $\boxed{\text{DIMREG}}$ (including UV and IR - divergent and finite terms)

Needed for HW 1.2.3 : $\boxed{b_{CT}} = \ominus b(m^2) + \left[2m^2a'(m^2)+2mb'(m^2)\right]$

$$= -(m\alpha/\pi)\left\{1/\varepsilon' - 1/2 - \int dx \log\left(x^2m^2/\mu^2\right)\right\}$$ [from $-b(m^2)$: use HW 1.2.2]

(DIMREG)

from ...a'...b' $\;\pm 2m^3\frac{\alpha}{2\pi}\int dx\left[(1-x)^2(-x)-2(-x)(1-x)\right]/x^2m^2$

$$\left[= \frac{\alpha m}{\pi}\left[-\frac{1}{\varepsilon'}+1/2+\int dx\log\left(\frac{x^2m^2}{\mu^2}\right)\right]+\frac{\alpha m}{\pi}\int\frac{dx}{x}(1-x)(1+x)\right]$$

$\boxed{2\cdot 2\cdot 1}$ $\boxed{(iii)}$ Using $\boxed{\text{cut-off}}$ regularization for $F_1(0)$: the UV-<u>finite</u> contribution is <u>same</u> as before, i.e., independent of UV-regulator ... UV-<u>divergent</u> contribution is given by 1st term in Eq. (11·81) of Lahiri and Pal, used in Eq. (11·80) ... but it turns out to be more convenient to do $\int dx_3$ instead of $\int dx_{1,2}$ (just like we discussed earlier for UV-finite contribution):

$$F_1(0) = 8\pi i\alpha \int \frac{d^4k_E}{(2\pi)^4} \underset{\substack{\uparrow \\ \text{due to going} \\ \text{to Euclidean momentum (denoted by } k_{Ⓔ})}}{ⓘ} (-1)^{1+3} \int dx_3 \frac{k_E^2}{\Big[k_E^2 + \underbrace{(1-x_3)^2}_{\text{was } (x_1+x_2)^2 \text{ earlier}} m^2\Big]^3}$$

$$\left.\times\ (1-x_3)\ \right\}\ \text{coming from (trivial) } \int dx_{1,2} \text{ (see earlier UV-finite calculation)}$$

The momentum integral is given by

$$\underbrace{\frac{2\pi^2}{\Gamma(2)}}_{\substack{\text{angular integration} \\ \text{see Eqs. (A·37)} \\ \text{\& (A·43) of} \\ \text{Lahiri \& Pal}}} \frac{1}{2}\int_0^{\Lambda^2_{UV}/a^2} dy \frac{y^2}{(y+1)^3}, \quad \text{where } a^2 = (1-x_3)^2 m^2 \text{ and } y = k_E^2/a^2$$

$$= \pi^2 \int_0^{\Lambda^2_{UV}/a^2} dy \frac{[(y+1)^2 - 2(y+1) + 1]}{(y+1)^3}$$

$$= \pi^2 \left\{ \log\left(\Lambda^2_{UV}/a^2 + 1\right) + 2\left[\frac{1}{\Lambda^2_{UV}/a^2+1} - \frac{1}{1}\right] - 1/2\left[\frac{1}{(\Lambda^2_{UV}/a^2+1)^2} - 1/1\right] \right\}$$

Neglect terms which → 0 as $\Lambda_{UV} \to \infty$, e.g.,

$$\log\left(\Lambda_{UV}^2/a^2 + 1\right) = \log \Lambda_{UV}^2/a^2 + \underbrace{\log\left(1 + \frac{a^2}{\Lambda_{UV}^2}\right)}_{\simeq a^2/\Lambda_{UV}^2 \to 0 \ldots}$$

$$= \pi^2\left[\log \frac{\Lambda_{UV}^2}{(1-x_3)^2 m^2} - 3/2\right]$$

so that

$$F_1(0)\ (\text{UV-divergent contribution}) = -\frac{\alpha}{2\pi} \times \Bigg\{ \underbrace{\int dx_3\,(1-x_3)}_{\left(x_3 - \frac{1}{2}x_3^2\right)\big|_0^1} \left[\log\left(\Lambda_{UV}^2/m^2\right) - 3/2\right] - 2\int_0^1 dx_3\, \underbrace{(1-x_3)}_{\tilde{x}_3} \log(1-x_3) \Bigg\}$$

$$= -\frac{\alpha}{2\pi}\left\{\frac{1}{2}\log\frac{\Lambda_{UV}^2}{m^2} - 3/4 - 2\left[\underbrace{\left(\frac{\tilde{x}_3^2}{2}\log \tilde{x}_3\right)\Bigg|_0^1}_{0} - \underbrace{\int_0^1 d\tilde{x}_3\, \frac{\tilde{x}_3^2}{2}\,\frac{1}{\tilde{x}_3}}_{1/4}\right]\right\}$$

$$= -\frac{\alpha}{2\pi}\left\{\frac{1}{2}\log\frac{\Lambda_{UV}^2}{m^2} - 1/4\right\}$$

Combining with UV-finite contribution calculated earlier, i.e., $F_1(0)\ (\text{UV-finite}) = -\frac{\alpha}{2\pi}\left(5/2 + \log\frac{m_\gamma^2}{m^2}\right)$

gives

$\boxed{F_1(0)}$ in <u>cut-off</u> regularization

$$= -\frac{\alpha}{2\pi}\left\{\frac{1}{2}\log\frac{\Lambda_{UV}^2}{m^2} + \log\frac{m_\gamma^2}{m^2} + \enclose{circle}{9/4}\right\}$$

$\boxed{2.2.2}$ $\boxed{(ii)}$ $(Z_2 - 1)$ using cut-off regularization:

again, focus only on UV-divergent contribution,

i.e., $a(m^2) = 2ie^2 \int_0^1 dx \, \textcircled{i} \int \frac{d^4k_E}{(2\pi)^4} \frac{(1-x)\,(-1)^2}{(k_E^2 + x^2m^2)^2}$

[use 1st line of Eq. 12.25 of LP]

going to Euclidean...

$$= -2 \cdot \frac{4\pi\alpha}{16\pi^4} \cdot \underbrace{2\pi^2}_{\text{angular...}} \int_0^1 dx \, \frac{1}{2} \int_0^{\Lambda^2_{UV}/a^2} \frac{dy \; y(1-x)}{(y+1)^2}, \text{ where } y = \frac{k_E^2}{a^2} \; \& \; a^2 = x^2m^2$$

$$= -\frac{\alpha}{2\pi} \int_0^1 dx\,(1-x) \int_0^{\Lambda^2_{UV}/a^2} \frac{dy\,[(y+1)-1]}{(y+1)^2}$$

$$= -\frac{\alpha}{2\pi} \int dx\,(1-x) \left\{ \log\left(\Lambda^2_{UV}/a^2 \cancel{+1}\right) + \left[\underbrace{\frac{1}{\Lambda^2_{UV}/a^2 + 1}}_{\to 0 \text{ as } \Lambda_{UV} \to \infty} - \frac{1}{1} \right] \right\}$$

drop as $\Lambda_{UV} \to \infty$

$$= -\frac{\alpha}{2\pi} \left\{ \left(x - \frac{x^2}{2}\right)\Big|_0^1 \times \left[\log\left(\frac{\Lambda^2_{UV}}{m^2}\right) - 1 \right] - 2\int_0^1 dx\,(1-x)\log x \right\}$$

$$= -\frac{\alpha}{2\pi} \left\{ \frac{1}{2} \log \Lambda^2_{UV}/m^2 - 1/2 - 2(x\log x - x)\Big|_0^1 + 2\left[\left(\frac{x^2}{2}\log x\right)\Big|_0^1 - \int_0^1 dx \, x^2/2 \, \frac{1}{x} \right] \right\}$$

$$= -\frac{\alpha}{2\pi} \left\{ \frac{1}{2} \log \Lambda^2_{UV}/m^2 \underbrace{- \frac{1}{2} + 2 - 2 \cdot 1/4}_{+1} \right\}$$

Adding UV-finite contributions, i.e., $2m^2 \frac{da}{dp^2}\Big|\ldots$ (i.e., same as in DIMREG) from earlier, gives

$$\underset{\text{in } \underline{\text{cut-off}}}{(Z_2 - 1)} = -\frac{\alpha}{2\pi}\left\{\frac{1}{2}\log\frac{\Lambda_{UV}^2}{m^2} + \log\frac{m_\gamma^2}{m^2} + \boxed{2}\right\}$$

$\boxed{2.2.3}\boxed{(ii)}$

$\Rightarrow$ UV and IR - <u>divergent</u> terms in $(Z_1 - 1)$ and $(Z_2 - 1)$ <u>are</u> equal, but not the finite one...

$$\left[(Z_1 - Z_2) \text{ with hard cut-off} = -\frac{\alpha}{2\pi}\left(9/4 - 2\right) = -\frac{\alpha}{8\pi} \right]$$

2.1: LP Ex. 12.8

Use Eq. (12.111) of Lahiri & Pal for every fermion to get

$$\alpha\left[s = 80\,(\text{GeV})^2\right] \simeq \frac{1}{137.03} + \left(\frac{1}{137.03}\right)^2 \frac{1}{3\pi} \times$$

$$\Bigg\{\left(-5/3\right)\Bigg[\underset{e,\mu,\tau}{3} + \overset{\text{color}}{3}\,\underset{\text{electric charge}}{\left(\frac{1}{3}\right)^2}\cdot \underset{d,s,b}{3} + 3\cdot \underset{u,c}{2}\cdot\left(2/3\right)^2 \ldots \underline{\text{no}}\text{ top quark since it's heavier than 80 GeV}\Bigg]$$

(As discussed in lecture, we get a factor of Q^2 from the loop, i.e., Q at each vertex ...)

$$+ \log\left(\frac{80\text{ GeV}}{0.511\text{ MeV}}\right)^2 (\text{for electron}) + \log\left(\frac{80}{0.106}\right)^2 + \log\left(\frac{80}{1.78}\right)^2$$

$$+\, 3\cdot 4/9\left[\log\left(\frac{80}{5\times10^{-3}}\right)^2 (\text{for up-quark}) + \log\left(\frac{80}{1.4}\right)^2\right]$$

$$+\, 3\cdot 1/9\left[\log\left(\frac{80}{9\times10^{-3}}\right)^2 (\text{for down-quark}) + \log\left(\frac{80}{0.17}\right)^2 + \log\left(\frac{80}{4.4}\right)^2\right]\Bigg]\Bigg\}$$

$$\simeq 1/(128.8) \simeq 7.76\times10^{-3}$$