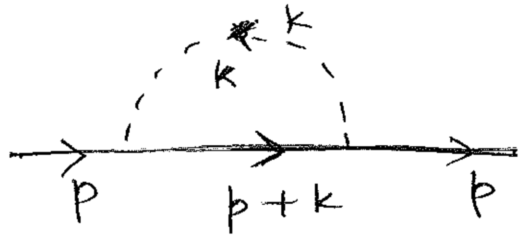
HW 1 | solutions

1.1 The fermion self-energy from scalar loop:

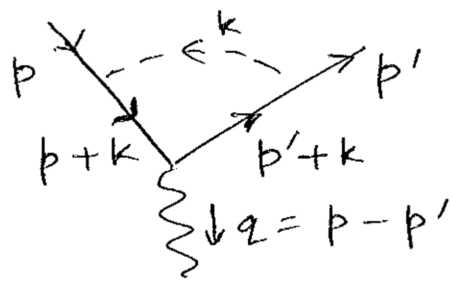
LP Ex. 12.2



fermion propagator

$$-i\Sigma^{(1)}(p) = \underbrace{(-ih)^2}_{\text{scalar-fermion vertex}} \int \frac{d^4k}{(2\pi)^4} \frac{i}{(\not{p}+\not{k}-m)} \underbrace{\frac{i}{k^2-m^2}}_{\text{scalar propagator}} \quad \ldots(1)$$

The QED vertex function from scalar loop:



$$-ie\,\Gamma_\mu^{(1)} = \underbrace{(-ieQ)}_{\text{photon-fermion vertex}} (-ih)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(\not{p}'+\not{k}-m)} \gamma_\mu \frac{i}{(\not{p}+\not{k}-m)} \times \frac{i}{(k^2-m^2)}$$

so that

$$q^\mu (-ie)\Gamma_\mu^{(1)} = (-ieQ)(-ih)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(\not{p}'+\not{k}-m)} \overbrace{\left[(\not{p}+\not{k}-m)-(\not{p}'+\not{k}-m)\right]}^{=\not{q}} \times \frac{i}{\not{p}+\not{k}-m} \frac{i}{k^2-m^2}$$

(similar to Lahiri & Pal, p. 254-255)

i.e., $$q^\mu \Gamma_\mu^{(1)} = (-ih)^2 Q \int \frac{d^4k}{(2\pi)^4} \left[\frac{i}{(\not{p}'+\not{k}-m)} - \frac{i}{\not{p}+\not{k}-m} \right] \times i \times \frac{i}{k^2-m^2}$$

$$= Q \left[\Sigma^{(1)}(p') - \Sigma^{(1)}(p) \right]$$ using Eq. (1) above

... just like the case of photon loop...

$\boxed{\text{LP Ex 12.5}}$ ✗ (HW 1.2.1 solutions are towards end)

$\boxed{1.2.2}$ Using dimensional regularization (DIMREG) in Eq. 12.21 of Lahiri & Pal gives

$$\Sigma(p) = -i\mu^{2\varepsilon} e^2 \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \times \frac{\gamma_\mu(\not{p}+\not{k}+m)\gamma^\mu}{[(p+k)^2-m^2]\,k^2}$$

Use the contraction formulae: $\gamma_\mu \gamma_\lambda \gamma^\mu = -2(1-\varepsilon)\gamma_\lambda$ in 1st term of numerator and $\gamma_\mu\gamma^\mu = (4-2\varepsilon)$ in 2nd term to find

$$= +i\mu^{2\varepsilon} e^2 \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \frac{2(1-\varepsilon)(\not{p}+\not{k}) - 4\left(1-\frac{\varepsilon}{2}\right)m}{[(p+k)^2-m^2]\,k^2}$$

Introducing Feynman parameter

$$= ie^2\mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \int_0^1 dx \frac{2(1-\varepsilon)(\not{p}+\not{k}) - 4m\left(1-\frac{\varepsilon}{2}\right)}{[k^2+2x\,k\cdot p + x(p^2-m^2)]^2}$$

Change variables : $\bar{k} = k + xp$

$$= i e^2 \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon} \bar{k}}{(2\pi)^{4-2\varepsilon}} \int_0^1 dx \; \frac{2(1-\varepsilon)\not{p} + 2(1-\varepsilon)(\bar{k} - x\not{p}) - 4m(1-\frac{\varepsilon}{2})}{\left\{\bar{k}^2 - [x(x-1)p^2 + x m^2]\right\}^2}$$

Drop the term linear in $\bar{k}$ in numerator (since denominator is even in $\bar{k}$ and we are integrating over all values of $\bar{k}$)

Matching above expression to $a(p^2)\not{p} + b(p^2)$ (as in Eq. 12.24 of Lahiri & Pal) gives

$$\boxed{a(p^2)} = i e^2 \mu^{2\varepsilon} \int_0^1 dx \; 2(1-\varepsilon)(1-x) \underbrace{\frac{i(-1)^2}{(4\pi)^{2-\varepsilon}} \frac{\Gamma[2-(2-\varepsilon)]/\Gamma(2)}{[m^2 x - x(1-x)p^2]^{\varepsilon}}}_{\text{with } r=0,\, s=2}$$

(where Eq. A.54 of Lahiri & Pal is used for the momentum integration

$$= -\frac{e^2}{16\pi^2} \int dx \; 2(1-x) \Big\{ \overbrace{\frac{1}{\varepsilon} - \gamma_E}^{\text{use } \Gamma(\varepsilon) = 1/\varepsilon - \gamma_E} - 1 + \log 4\pi$$

$$- \log[x m^2 - x(1-x)p^2] + \log \mu^2 \Big\}$$

from $\frac{1}{\varepsilon}$ from $\Gamma(\varepsilon)$ hitting $(1-\varepsilon)$ outside

from $\frac{1}{(4\pi)^{-\varepsilon}} \approx (1 + \varepsilon \log 4\pi)$ and $1/\varepsilon \cdots$

from $\frac{1}{[m^2 x \ldots]^{\varepsilon}} \cdots$

from $\mu^{2\varepsilon} \cdots$

$$= -\frac{\alpha}{2\pi} \left\{ \underbrace{\int_0^1 dx\,(1-x)}_{\text{gives } 1/2} \left(1/\varepsilon - 1 - \gamma_E + \log 4\pi \right) - \int_0^1 dx\,(1-x) \log [x m^2 - x(1-x)p^2]/\mu^2 \right\}$$

$$= \frac{\alpha}{2\pi}\left\{\frac{1}{2} - \frac{1}{2\varepsilon'}\right.$$ $1/\varepsilon' = 1/\varepsilon - \gamma_E + \log 4\pi$ as in Lahiri & Pal, Eq. 12.55 (4)

$$\left. + \int_0^1 dx\,(1-x)\log\left[\frac{x m^2 - x(1-x)p^2}{\mu^2}\right]\right\}$$ as in Lahiri & Pal, Eq. 12.68

and $\boxed{b/m} = i e^2 \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \int_0^1 dx \frac{-4(1-\varepsilon/2)}{\{k^2 - [x m^2 - x(1-x)p^2]\}^2}$

$$= i e^2 \frac{i}{16\pi^2}(-4)\int_0^1 dx \left\{\frac{1}{\varepsilon} - \gamma_E + \log 4\pi - 1/2 \leftarrow \text{from } \left(1-\tfrac{\varepsilon}{2}\right)\ldots\right.$$

$$\left. - \log[x m^2 - x(1-x)p^2]/\mu^2\right\}$$

$$= \frac{\alpha}{\pi}\left\{\frac{1}{\varepsilon'} - 1/2 - \int dx \log[x m^2 - x(1-x)p^2]/\mu^2\right\}$$

again in agreement with Lahiri & Pal, Eq. 12.68

$\boxed{1.2.3}$ $\boxed{\text{LP Ex. 12.7}}$ use above result

$a(p^2) + a_{CT}$ in DIMREG

use results shown in solutions to HW $\boxed{2.2.2}$ (i)

$$= \frac{\alpha}{2\pi}\left\{-\frac{1}{2\varepsilon'} + \frac{1}{2} + \int dx\,(1-x)\log\frac{[x m^2 - x(1-x)p^2]}{\mu^2}\right\}$$

$$+ \frac{\alpha}{2\pi}\left[+\frac{1}{2\varepsilon'} - \frac{1}{2} - \int dx\,(1-x)\log\frac{x^2 m^2}{\mu^2}\right]$$ from $\ominus a(m^2)$

$$+ \frac{\alpha}{\pi}\int \frac{dx}{x}(1-x)(-1-x)$$ from $\ominus 2m^2 a'(m^2)$ $\ominus 2m\, b'(m^2)$

$$= \frac{\alpha}{2\pi}\int dx\,(1-x)\left\{\log\frac{[x m^2 - x(1-x)p^2]}{x^2 m^2} - 2\frac{(1+x)}{x}\right\}$$

i.e., dependence on ε and μ^2 disappears in $a(p^2) + a_{CT}$ which is coefficient of $\not{p}$ in Σ_R

Similarly, coefficient of $\mathbb{1}$ in Σ_R, i.e.,

$$\boxed{b(p^2) + b_{CT}} = -\frac{\alpha m}{\pi}\left\{-\frac{1}{\varepsilon'} + \frac{1}{2} + \int_0^1 dx \log\frac{[x m^2 - x(1-x)p^2]}{\mu^2}\right\}$$

$$+\frac{\alpha m}{\pi}\left[-\frac{1}{\varepsilon'} + \frac{1}{2} + \int_0^1 dx \log\frac{x^2 m^2}{\mu^2}\right]\Big\} \text{ from } -b(m^2$$

$$+\frac{\alpha m}{\pi}\int\frac{dx}{x}(1-x)(1+x)\Big\} \text{ from } +2m^3 a'(m^2) + 2m^2 b'(m^2)$$

$$= -\frac{\alpha m}{\pi}\int dx\left\{\log\frac{[x m^2 - x(1-x)p^2]}{x^2 m^2} - \frac{(1-x^2)}{x}\right\}$$

i.e., again no dependence on $\boxed{\varepsilon \text{ or } \mu^2}$ ... [last part of HW 1.2.1]

Using Pauli-Villars regularization, we get

$\boxed{a(p^2) + a_{CT}}$ [combining Eqs. (12.67) and (12.87) of Lahiri & Pal]

$$= \frac{\alpha}{2\pi}\int_0^1 dx\,(1-x)\left\{\log\frac{\left[\frac{x m^2 - x(1-x)p^2}{(1-x)M^2}\right]}{[x^2 m^2/(1-x)M^2]} - \frac{2(1+x)}{x}\right\}$$

$$= \frac{\alpha}{2\pi}\int_0^1 dx\,(1-x)\left\{\log\frac{[x m^2 - x(1-x)p^2]}{x^2 m^2} - 2\left(\frac{1+x}{x}\right)\right\}$$

i.e., dependence on M^2 disappears in $\left[a_\Lambda(p^2) + a_{CT}\right]$ ...

... and $a_\Lambda(p^2) + a_{CT}$ is <u>same</u> in DIMREG vs. Pauli-Villars regularization

Similarly, $\boxed{b(p^2) + b_{CT}}$ in Pauli-Villars regularization

$$= -\frac{\alpha m}{\pi} \int_0^1 dx \left\{ \log \frac{\left[\frac{x m^2 - x(1-x) p^2}{(1-x) M^2}\right]}{\left[x^2 m^2 / (1-x) M^2\right]} - \frac{(1-x^2)}{x} \right\}$$

$$= -\frac{\alpha m}{\pi} \int_0^1 dx \left\{ \log \left[\frac{x m^2 - x(1-x) p^2}{x^2 m^2}\right] - \frac{(1-x^2)}{x} \right\}$$

i.e., again <u>in</u>dependent of $\boxed{M^2}$ and <u>agrees</u> with the <u>DIMREG</u> result...

—— × ——

The $\boxed{\text{total}}$ 2-point function, i.e., tree + 1 loop + CT, is $\not{p} - m - \Sigma_R(p)$

$$= \not{p} - m - \left[a(p^2) + a_{CT}\right] \not{p} + \left[b(p^2) + b_{CT}\right]$$

So, since $\boxed{a + a_{CT}}$ is <u>same</u> in DIMREG vs. Pauli-Villars regularization, it is clear that the <u>total</u> 2-point function also is the <u>same</u> (as expected, since this is the $\boxed{\text{physical}}$ quantity)

1.2.1 LP Ex. 12.6

Use $\Sigma_R(p) = [a(p^2) + a_{CT}]\not{p} + b(p^2) + b_{CT}$

no p dependence here

and $p^2 = \not{p}\not{p}$ to give

$$\frac{\partial \Sigma_R(p)}{\partial \not{p}}\Bigg|_{\underbrace{\not{p}=m}_{\Rightarrow p^2=m^2}} = a(p^2=m^2) + \not{p}\frac{da(p^2)}{d\not{p}}\Bigg|_{\not{p}=m} + a_{CT} + \frac{db(p^2)}{d\not{p}}\Bigg|_{\not{p}=m}$$

$$= a(m^2) + \frac{da(p^2)}{dp^2}\frac{dp^2}{d\not{p}}\Bigg|_{\not{p}=m} \times m + a_{CT} + \frac{db(p^2)}{dp^2}\frac{dp^2}{d\not{p}}\Bigg|_{\not{p}=m}$$

$$= a(m^2) + a'(m^2)(2\not{p})\Big|_{\not{p}=m} \times m + a_{CT} + b'(m^2)(2\not{p})\Big|_{\not{p}=m}$$

$$= a(m^2) + 2m^2a'(m^2) + a_{CT} + 2m\,b'(m^2)$$

So, the condition $\frac{\partial \Sigma_R(p)}{\partial \not{p}}\Big|_{\not{p}=m} = 0$ fixes

$$\boxed{a_{CT} = -a(m^2) - 2m^2a'(m^2) - 2m\,b'(m^2)}$$

Eq. 12.85 of LP

as given in Eq. 12.86 of LP

And, the other condition $\Sigma_R(p)\big|_{\not{p}=m} = 0$ gives

$$[a(m^2) + a_{CT}]m + b(m^2) + b_{CT} = 0$$

so that (using above a_{CT})

$$\boxed{b_{CT} = -b(m^2) + 2m^3a'(m^2) + 2m^2\,b'(m^2)}$$

as given in Eq. 12.86 of LP

→ These are valid in $\boxed{\text{any}}$ regularization scheme

Note : solutions to last part, i.e., total 2-point function in PV are given at end of HW 1.2.3 (8)

In Pauli-Villars regularization, $a(p^2)$ & $b(p^2)$ are given / derived in Eq. 12.67 of LP :

plugging these in Eq. 12.86 of LP (i.e., above a_{CT}, b_{CT}) gives

$$a_{CT} = -\frac{\alpha}{2\pi}\int_0^1 dx \Big[(1-x)\ln\left(\frac{x^2m^2}{(1-x)M^2}\right) \leftarrow \text{from } a(m^2)$$

$$+ 2\cancel{m^2}\frac{(-x)(1-x)^2}{x^2\cancel{m^2}} \leftarrow \text{from } a'(m^2)$$

$$-2.2\cancel{m^2}\frac{(-x)(1-x)}{x^2\cancel{m^2}} \leftarrow \text{from } b'(m^2)\Big]$$

$$= -\frac{\alpha}{2\pi}\int_0^1 dx\,(1-x)\left[\ln\left(\frac{x^2m^2}{(1-x)M^2}\right) + \frac{2}{x^2}(-x+x^2+2x)\right]$$

i.e., $$\boxed{a_{CT}} = -\frac{\alpha}{2\pi}\int_0^1 dx\,(1-x)\left[\ln\left(\frac{x^2m^2}{(1-x)M^2}\right) + \frac{2}{x}(x+1)\right]$$

as given in Eq. 12.87 of LP (with $x \to \xi$)

Similarly,

$$b_{CT} = \frac{\alpha}{\pi}\int_0^1 dx \Big[m\ln\left(\frac{x^2m^2}{(1-x)M^2}\right) \leftarrow \text{from } b(m^2)$$

$$+ \cancel{2}m^3\frac{1}{\cancel{2}}\frac{(-x)(1-x)^2}{x^2m^2} \leftarrow \text{from } a'(m^2)$$

$$+ 2m^3\frac{x(1-x)}{x^2m^2} \leftarrow \text{from } b'(m^2)\Big]$$

i.e., $$\boxed{b_{CT}} = \frac{\alpha m}{\pi}\int_0^1 dx\left[\ln\left(\frac{x^2m^2}{(1-x)M^2}\right) + \frac{1-x^2}{x}\right]$$ as given in Eq. 12.87 of LP