Any ambiguity in solving this equation is to be resolved by rotating to imaginary time.

Note that if S' contains derivatives of the qs, these will just become derivatives of propagators in the expansion. The familiar problem of pushing time derivatives of quantum fields through a time-ordering operator, the problem that makes perturbation theory for derivative interactions such a combinatoric nightmare, has no counterpart here, for we have no time-ordering operator and no quantum fields, just an integral over c-number fields.

Thus, for any theory, if we can write the generating functional in the form (4.49), we can just read off the Feynman rules from S' in the most naive way, replacing every derivative of a field with a momentum factor, etc., without making any mistakes. Unfortunately, at the moment, the only theories for which we can write the generating functional in the form (4.49) are those without any derivatives in the interaction, so this observation is without immediate use. However, it will become very useful shortly.

4.4 *Derivative interactions*

There is a large class of theories with derivative interactions for which it is possible to write a functional-integral representation of the generating functional. These are theories where the Lagrangian is no more than quadratic in time derivatives,

$$L=\tfrac{1}{2}\dot{q}^a K_{ab}\dot{q}^b + L_a\dot{q}^a - U, \tag{4.53}$$

where K, L, and U are functions of the qs. The only restriction I will place on these functions is that K be invertible, so that the equation for the canonical momenta,

$$p_a = K_{ab}\dot{q}^b + L_a, \tag{4.54}$$

can be solved for the $\dot{q}$s and the Hamiltonian constructed,

$$H=\tfrac{1}{2}p_a(K^{-1})^{ab}p_b + \cdots, \tag{4.55}$$

where the triplet dots indicate terms of first and zeroth order in the ps.

For these theories, the appropriate generalization of our earlier result, Eq. (4.42), turns out to be

$$e^{iW} = N\int \prod_a (dq^a)[\det K]^{\frac{1}{2}} e^{iS}. \tag{4.56}$$

In this equation, K is to be interpreted as a linear operator on the function space, and the integral is to be interpreted in the same way our earlier (Gaussian) integrals were interpreted. Everything is to be restricted to a finite-dimensional subspace, the integral is to be done over that subspace, and the limit is to be taken. I do not know of any short argument for this formula, and have to refer you to the literature for a proof.[21] However, I can try and make it plausible to you by showing that it obeys some simple consistency checks. (1) If K is independent of the qs, and L vanishes, this reduces to the previous case. The determinant can then be pulled out of the integral and absorbed by the normalization factor, reproducing Eq. (4.42). (2) If K is independent of the qs, but L does not vanish, then, by our earlier remarks, the Feynman rules are the naive ones, with the derivative in the interaction becoming a factor of momentum at the vertex. This may be a familiar result to you if you have ever gone through the derivation of the Feynman rules for ps-pv meson–nucleon theory, or the electrodynamics of charged scalar bosons. (3) If K does depend on the qs, things are not so simple. This may be familiar to you if you followed the discussion in the literature a few years ago about the Feynman rules for chiral Lagrangians.

(4) Finally, a Lagrangian of the form (4.48) becomes one of the same form if we change coordinates. To be more precise, let us trade the qs for new variables, which we denote by $\bar{q}^a$. Then

$$L=\tfrac{1}{2}\dot{q}^a K_{ab}\dot{q}^b + \cdots = \tfrac{1}{2}\dot{\bar{q}}^a \bar{K}_{ab}\dot{\bar{q}}^b + \cdots, \tag{4.57}$$

where

$$\bar{K}_{ab} = \frac{\partial q^d}{\partial \bar{q}^a} K_{cd} \frac{\partial q^d}{\partial \bar{q}^b}. \tag{4.58}$$

This takes care of the transformation of the Lagrangian, but we still have to change variables in the functional integral. As always, we will figure out how to do this by going back to the finite-dimensional case. Suppose, in a finite dimensional space, we change from coordinates $\mathbf{x}$ to coordinates $\bar{\mathbf{x}}$. Even though $\bar{\mathbf{x}}$ may be a non-linear function of $\mathbf{x}$, $\partial\mathbf{x}/\partial\bar{\mathbf{x}}$ is a linear operator (an $n\times n$ matrix, where n is the dimension of the space), and has a determinant. The change-of-variables formula is the familiar Jacobian formula,

$$(d\mathbf{x}) = (d\bar{\mathbf{x}}) \det(\partial\mathbf{x}/\partial\bar{\mathbf{x}}). \tag{4.59}$$

As always, we simply extend this to the infinite-dimensional case, obtaining

$$\begin{aligned}[\det K]^{\frac{1}{2}} \prod_a (dq^a) &= [\det K]^{\frac{1}{2}} \prod_a (d\bar{q}^a)\det(\partial q/\partial\bar{q}) \\ &= \prod_a (d\bar{q}^a)[\det \bar{K}]^{\frac{1}{2}}. \end{aligned} \tag{4.60}$$

Thus, Eq. (4.56) is independent of our choice of coordinates.

Eq. (4.56) is sometimes written in 'Hamiltonian form',[22]

$$e^{iW} = N\int \prod_a (dp^a)(dq_a) e^{iS}, \tag{4.61}$$

4.6 *Ghost fields*

We left the theory of derivative interactions in poor shape. It is true that we had an expression for the generating functional, Eq. (4.56), but it was not in the form of an integral of an exponential; there was a determinant sitting in front. Therefore, we could not use Eq. (4.56) to develop a diagrammatic perturbation expansion of the integral. We can now use our knowledge of Fermi fields to get the determinant up into the exponential. For, if we introduce a set of complex Fermi variables, η^a, and denote by $K^{\frac{1}{2}}$ the matrix square-root of K, then

$$[\det K]^{\frac{1}{2}} = \int (\mathrm{d}\eta^*)(\mathrm{d}\eta)\mathrm{e}^{\mathrm{i}(\eta^{a*}, K^{\frac{1}{2}}_{ab}\eta^b)}, \tag{4.68}$$

up to a multiplicative constant, which can always be absorbed in the normalization factor, N. The ηs are called ghost variables (in the field-theory case, ghost fields). They are not true dynamical variables of the system, simply devices for getting a determinant up into an exponential.

Thus, the Feynman rules for the theory can be read off from an 'effective Lagrangian',

$$L_{\text{eff}} = L + L_{\text{g}}, \tag{4.69}$$

where L_{g}, the ghost Lagrangian, is given by

$$L_{\text{g}} = \eta^{*a}K^{\frac{1}{2}}_{ab}\eta^b. \tag{4.70}$$

It is instructive to work out in detail a field-theoretic example. Let us consider the theory of a free field coupled to an external source,

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + J\phi. \tag{4.71}$$

Let us make a change of variables to a new field, A, defined by

$$\phi = A + \tfrac{1}{2}gA^2. \tag{4.72}$$

where g is a constant. (This transformation is not invertible, but that shouldn't worry us; we're only going to do perturbation theory, and (4.72) is invertible near $\phi=0$.) In terms of A, the Lagrange density is given by

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu A)^2(1+gA)^2 - \tfrac{1}{2}u^2A^2(1+\tfrac{1}{2}gA)^2 + JA(1+\tfrac{1}{2}gA). \tag{4.73}$$

Thus we apparently have a very complicated interaction, with g some sort of coupling constant. Of course, this interaction is just an illusion; the vacuum-to-vacuum matrix element must be the same as in our original theory. However, this is not the answer you will get if you just read the Feynman rules naively out of (4.73). The right Feynman rules are obtained from an effective Lagrange density

$$\mathscr{L}_{\text{eff}} = \mathscr{L} + \mathscr{L}_{\text{g}}, \tag{4.74}$$

where

$$\mathscr{L}_{\text{g}} = \eta^*\eta(1+gA). \tag{4.75}$$

The unphysical nature of the ghost fields is doubly clear from this expression. (1) The ghost fields are spinless fields obeying Fermi statistics. (2) The ghost propagator has no momentum dependence; it is a constant, i.

I recommend that you compute a few things to low orders of perturbation theory, using this effective Lagrange density, to convince yourself that everything works out as it should. A good starting point is the one-point function (tadpole) to order g. This should vanish. Does it?

5 The Feynman rules for gauge field theories

5.1 *Troubles with gauge invariance*

The quantization of gauge field theories is notoriously tricky. We can get an idea of the problem if we look at the simplest gauge-invariant field theory, electromagnetism.

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \mathscr{L}'. \tag{5.1}$$

Let us try and derive the Feynman propagators for A_μ by straightforwardly applying the methods of Section 4, without worrying about whether electromagnetism is in fact in the class of theories we discussed there. The computation is simplified by splitting the field into (four-dimensional) transverse and longitudinal parts

$$\begin{aligned} A_\mu &= A^T_\mu + A^L_\mu \\ &= (P^T_{\mu\nu} + P^L_{\mu\nu})A^\nu. \end{aligned} \tag{5.2}$$

where the Ps are the transverse and longitudinal projection operators; in Fourier space they are given by

$$P^T_{\mu\nu} = g_{\mu\nu} - k_\mu k_\nu/k^2, \; P^L_{\mu\nu} = k_\mu k_\nu/k^2. \tag{5.3}$$

(Remember, we are secretly doing all our computations in Euclidean space, so there is no ambiguity in dividing by k^2.) Then it is easy to see that

$$S = \int \mathrm{d}^4x[\tfrac{1}{2}(\partial_\mu A^T_\nu)^2 + \mathscr{L}']. \tag{5.4}$$

We obtain the propagators for the transverse and longitudinal parts of the field by our standard formulae; thus

$$D^{\mathrm{F}}_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\left(-\frac{\mathrm{i}}{k^2}\right) + \frac{k_\mu k_\nu}{k^2}\left(\frac{i}{0}\right). \tag{5.5}$$

The second term is obviously unacceptable; something has gone wrong.

This debacle can be explained in two ways, either from Feynman's sum over histories or from conventional canonical quantization. (1) Sum-over-