$\boxed{\text{Notes}}$ of lectures on September 29, October 1 on Dirac equation

— Derivation of Dirac equation in $\boxed{2}$ "steps":

(1) Determine spin-$\frac{1}{2}$ representation of 4D Lorentz group (i.e, 3D rotations ⊕ boosts)

(2) Determine Lorentz invariant equation for "wavefunction" in above representation

(for alternate derivation, see Lahiri and Pal sections 4.1 & 4.2)

Step (1) Use analogy with 3D (spatial) rotations to figure out spin-$\frac{1}{2}$ representation of 4D Lorentz group

(for more "formal" derivation, see Peskin and Schroeder section 3.1 & 3.2 or Ryder section 2.3)

3D (spatial) rotations can be described by

new frame
$$\boxed{X^{\prime} = R^{+} X R}$$
old frame

where $X = x^i \sigma^i$ ($x^i = x, y, z$ and σ^i are Pauli matrices)

and $X^{\prime} = x^{\prime i} \sigma^i$

i.e., $X = \begin{bmatrix} x^3 & x^1 + i x^2 \\ x^1 - i x_2 & -x^3 \end{bmatrix}$, R is (thus far general) 2×2 matrix

We require the following

(i) $X^{\prime}$ is hermitian, just like X: that's already the case due to presence of R & $R^\dagger$ above

(ii) $\sum_{i=1}^{3} (x'^{i})^2 = \sum_{i=1}^{3} (x^i)^2$, i.e., $\underline{\underline{\det}}\, X' = \det X$

(i.e., length of $\bar{x}$ invariant)

$= \det X \det R^\dagger R$

$= \det X \, |\det R|^2$

$\Rightarrow \boxed{\det R = 1}$ up to phase (which is irrelevant due to $R^\dagger X R$)

So, R must be of the form

$\boxed{e^{z^i \sigma^i}}$ (where z^i are complex numbers)

since then $\det R = e^{z^i \underline{\underline{\mathrm{tr}}}\, \sigma^i} = e^0 = 1$

and σ^i's form a basis for 2×2 traceless matrices

(iii) X' is trace$\underline{\underline{\text{less}}}$, just like X : consider infinitesimal form of $R \simeq 1 + z^i \sigma^i$

so that

$$\mathrm{tr}\, X' \simeq \mathrm{tr}\left[\left(1 + z^{i*} \sigma^{i\dagger}\right) X \left(1 + z^i \sigma^i\right)\right]$$

$$\simeq \underset{\overset{\|}{0}}{\mathrm{tr}\, X} + \mathrm{tr}\left(z^{i*} \sigma^i x^j \sigma^j + x^j \sigma^j z^i \sigma^i\right)$$

$$= \mathrm{tr}\, \sigma^i \sigma^j \times x^j \underbrace{\left(z^{i*} + z^i\right)}_{2 \mathrm{Re}\, z} \overset{\text{require}}{=} 0$$

$\Rightarrow$ z^i must be purely imaginary

So, finally, we have $\boxed{R = e^{-i\theta^i \sigma^i/2}}$

— Of course $R_{2\times2}$ (or its infinitesimal version σ^i can also act (linearly) on 2 component column, i.e., implements rotations on "spinors" ... gives usual spin up & down states of non-relativistic QM (e.g. $\begin{pmatrix}1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\end{pmatrix}$ are eigenstates of σ^3 ...)

Onto 4D Lorentz group (spatial rotations + boosts)

In analogy with rotations above, use

$$X = \sum_{\mu=0}^{3} x^\mu \sigma^\mu \quad \text{with} \quad \sigma^\mu = \left(\mathbb{1}_{2\times2}, \sigma^{i=1,2,3}\right)$$

$$= \begin{bmatrix} \overset{\text{new}}{x^0} + x^3 & x^1 + i x^2 \\ x^1 - i x^2 & \underset{\text{new}}{x^0} - x^3 \end{bmatrix}$$

Again, Lorentz transformation is given by $\boxed{X' = L^\dagger X L}$ so that X' is hermitian (just like X) ... but now we only require $\det X' = \det X$

i.e., $x'^\mu x'_\mu = x^\mu x_\mu$ (where $x_\mu = g_{\mu\nu} x^\nu$)

(X' and X are not traceless)

$\Rightarrow$ general form of $L = e^{z^i \sigma^i}$ $z^i \rightarrow$ complex i.e., $\boxed{L = e^{-i\bar{\theta}\cdot\frac{\bar{\sigma}}{2} - \bar{\beta}\cdot\frac{\bar{\sigma}}{2}}}$ ($\bar{\theta}, \bar{\beta}$ are real)

(usual) spatial rotations — parametrize boosts

Again, L can implement Lorentz transformation (called LT from now on) on 2-component spinors

i.e. but if we specify $\bar{\theta}, \bar{\beta}$ based on LT of x^{μ}, then

$\boxed{L^{(\prime)} \; e^{-i\bar{\theta}\cdot\bar{\sigma}/2 \; \oplus \; \bar{\beta}\cdot\bar{\sigma}/2}}$ also implements LT on spinor

vs. "$-$" above

i.e., there are two types of 2-component spinors (called Weyl spinors): $\psi_{L,R}$

(where the notation L, R will be clear later)

which transform differently (under Lorentz group as $e^{-i\bar{\theta}\cdot\bar{\sigma}/2 \mp \bar{\beta}\cdot\bar{\sigma}/2}$ (respectively)

(i.e., same way under spatial rotations, but not boosts)

— Combine $\psi_{L,R}$ to form 4-component spinor (called Dirac spinor): $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$