Homework 8: Due November 3

1. Equation 3.3.21 in Sakurai gives an explicit matrix expression for a finite rotation specified the three Euler angles α, β , γ for a spin ½ system.
   a. Show that this rotation matrix may be represented as

$$\mathscr{D}(\alpha,\beta,\gamma) = \left(\cos(\beta/2)\cos((\alpha+\gamma)/2)\right)\overleftrightarrow{1} + i\left(\left(\sin(\beta/2)\sin((\alpha-\gamma)/2)\right)\overleftrightarrow{\sigma}_x - \sin(\beta/2)\cos((\alpha-\gamma)/2)\overleftrightarrow{\sigma}_y - \cos(\beta/2)\sin((\alpha+\gamma)/2)\overleftrightarrow{\sigma}_z\right)$$

   b. Instead of the three Euler angles one can parameterize a general rotation as the rotation about a fixed axis $\hat{n}$ through a fixed angle, θ.

Use the result in a. to show that

$$\vartheta/2 = \cos^{-1}\left(\cos(\beta/2)\cos((\alpha+\gamma)/2)\right)$$

$$\hat{n} = -\frac{\left(\sin(\beta/2)\sin((\alpha-\gamma)/2)\right)\hat{x} - \sin(\beta/2)\cos((\alpha-\gamma)/2)\,\hat{y} - \cos(\beta/2)\sin((\alpha+\gamma)/2)\hat{z}_z}{\sin(\vartheta/2)}$$

where the hat on *n,x, y*, and *z* indicates that they are unit vectors (as opposed to operators.

2. This problem is designed to fill in some gaps in our derivation of Schwinger's approach to angular momentum. Specifically, it concerns implementing finite rotations on the creation and annihilation operators.
   a. Using the definitions of the operators $\hat{\vec{J}}$ in terms of the creation and annihilation operators for +/- given in Eqs. 3.8.13 to show that $\left[(\hat{a}_+^+, \hat{a}_-^+), \hat{J}_k\right] = -(\hat{a}_+^+, \hat{a}_-^+)\frac{\overleftrightarrow{\sigma_k}}{2}$ where $\overleftrightarrow{\sigma_k}$ is a Pauli matrix and $(\hat{a}_+^+, \hat{a}_-^+)$ is a row vector. .
   b. From the result in a. show that

$$\exp\left(i\hat{n}\cdot\hat{\vec{J}}\theta\right)(\hat{a}_+^+, \hat{a}_-^+)\exp\left(-i\hat{n}\cdot\hat{\vec{J}}\theta\right) = (\hat{a}_+^+, \hat{a}_-^+)\left(\cos(\theta/2)\overleftrightarrow{1} + i\sin((\theta/2)\hat{n}\cdot\overleftrightarrow{\sigma_k}\right)$$

where $\overleftrightarrow{1}$ is the identity matrix. Hint: consider infinitesimal results and integrate

3. Using the fact that $\left|j,m\right\rangle = \frac{(a_+^+)^{j+m}(a_-^+)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left|0\right\rangle$ derive eq. 3.5.41 starting from Eqs. 3.8.13 and using only the commutation relations for the creation and annihilation operators for the harmonic oscillator.

4. Conventionally one quantizes the angular momentum in terms of eignenstates in the z-direction. In principle, one could take any direction. The purpose of this problem is to relate an eigenstate of $\hat{n}\cdot\hat{\vec{J}}$ (where $\hat{n}$ is an arbitrary unit vector) in terms of eigenstates in the z-direction. In principle this can be implemented straightforwardly using the Wigner –D matrices. In this problem, however, I want you to use Schwinger's approach directly.

a. As a first example, I want you to consider a state with fixed j (*i.e.* an eigenstates of $\hat{J}^2$ and of $\hat{n}\cdot\hat{\vec{J}}$ with eigenvalue of j (*i.e.* the maximum possible value) and express it as sum of states with good $|\,j,m\rangle$. That is, if we define $|\psi\rangle$ such that $\hat{J}^2|\psi\rangle = j(j+1)|\psi\rangle$ and $\hat{n}\cdot\hat{\vec{J}}|\psi\rangle = j|\psi\rangle$ and express $|\psi\rangle$ as $|\psi\rangle = \sum_m c_m\,|\,jm\rangle$, I want you to use Schwinger's approach to find the coefficients c_m.

b. As a second example, I want you to consider a state with fixed j (*i.e.* an eigenstates of $\hat{J}^2$ and of $\hat{n}\cdot\hat{\vec{J}}$ with eigenvalue of j-1 (*i.e.* one les than the the maximum possible value) and express it as sum of states with good $|\,j,m\rangle$. That is, if we define $|\Phi\rangle$ such that $\hat{J}^2|\Phi\rangle = j(j+1)|\Phi\rangle$ and $\hat{n}\cdot\hat{\vec{J}}|\Phi\rangle = (j-1)\Phi$ and express $|\Phi\rangle$ as $|\Phi\rangle = \sum_m d_m\,|\,jm\rangle$, I want you to use Schwinger's approach to find the coefficients d_m.

Sakurai----3.2, 3.4