Solution to PHYS606 Problem Set 3

February 16, 2011

[1]

1 Problem 1: Angular momentum

1.1 a)

Using

$$\partial_\mu T^{\mu\nu} = -\frac{1}{c} F^{\nu\lambda} J_\lambda \tag{1}$$

We have

$$\begin{aligned}
&\partial_\lambda M^{\mu\nu\lambda} = \partial_\lambda (x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}) \\
&= \delta^\mu_\lambda T^{\nu\lambda} - \delta^\nu_\lambda T^{\mu\lambda} + x^\mu \partial_\lambda T^{\nu\lambda} - x^\nu \partial_\lambda T^{\mu\lambda} \\
&= x^\mu \partial_\lambda T^{\nu\lambda} - x^\nu \partial_\lambda T^{\mu\lambda} \\
&= \frac{1}{c}(x^\nu F^{\mu\alpha} J_\alpha - x^\mu F^{\nu\rho} J_\rho) \\
&= \frac{1}{c}(x^\mu J_\alpha F^{\alpha\nu} - x^\nu J_\alpha F^{\alpha\mu})
\end{aligned}$$

1.2 b)

The conserved "current" corresponding to rotation is $M^{\mu\nu\lambda}$, which means that

$$\partial_\lambda M^{\mu\nu\lambda} = 0 \tag{2}$$

in the absence of external currents that are coupled to the field.

In general, for a conserved "current" j^νcorresponding to a continuous symmetry of a system, one has

$$\partial_0 \int d^3x j^0 = -\int d^3x \partial_i j^i = -\int d\vec{S} \cdot \vec{j} = 0 \tag{3}$$

[1]This set of solutions is partly adapted from Vivek Saxena's.

if we set the region of integration to be large enough so that the surface term vanishes.

This means that

$$Q^0 = \int d^3x j^0 \tag{4}$$

is a conserved quantity, usually termed "conserved charge".

Specifically for $M^{\mu\nu\lambda}$, we have

$$Q_M^{\mu\nu} = \int d^3x M^{\mu\nu 0} \tag{5}$$

are the conserved charges corresponding to rotation.

Using the expression of $T^{\mu\nu}$

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \frac{1}{c}S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{1}{c}S_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{1}{c}S_z & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix} \tag{6}$$

in which

$$\mathcal{E} = \frac{1}{8\pi}(|\vec{E}|^2 + |\vec{B}|^2) \tag{7}$$

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} \tag{8}$$

Therefore

$$M^{ij0} = x^i T^{j0} - x^j T^{i0} = \frac{1}{c}(x^i S^j - S^i x^j) = \frac{1}{c}\epsilon^{ijk}(\vec{x} \times \vec{S})_k \tag{9}$$

The corresponding conserved charge is

$$Q_M^{ij} = \int d^3x \frac{1}{c}\epsilon^{ijk}(\vec{x} \times \vec{S})_k \tag{10}$$

$$M^{i00} = x^i T^{00} - x^0 T^{i0} = x^i \mathcal{E} - x^0 \frac{1}{c} S^i = (\vec{x}\mathcal{E} - \frac{t}{c}\vec{S})^i \tag{11}$$

The corresponding conserved charge is

$$Q_M^{i0} = \int d^3x (\vec{x}\mathcal{E} - \frac{t}{c}\vec{S})^i \tag{12}$$

2 Problem 2: Stress tensor in action

2.1 a)

Assuming that the two conducting planes are located at planes $x = 0$ and$x = 0$, then the electric field between them is

$$\vec{E} = -4\pi\sigma\vec{e_x}(0 < x < a) \tag{13}$$

2.2 b)

The potential is

$$\varphi(x) = -4\pi\sigma x(0 < x < a) \tag{14}$$

2.3 c)

The electric field generated by the charged planes in the left is $2\pi\sigma\vec{e_x}$,therefore the force (per unit area) it exerts on the other plane is

$$\vec{f} = -2\pi\sigma^2\vec{e_x} \tag{15}$$

2.4 d)

The stress tensor is the spatial part of stress-energy tensor

$$T^{ij} = \begin{pmatrix} -\frac{1}{8\pi}|\vec{E}|^2 & 0 & 0 \\ 0 & \frac{1}{8\pi}|\vec{E}|^2 & 0 \\ 0 & 0 & \frac{1}{8\pi}|\vec{E}|^2 \end{pmatrix} = \begin{pmatrix} -2\pi\sigma^2 & 0 & 0 \\ 0 & 2\pi\sigma^2 & 0 \\ 0 & 0 & 2\pi\sigma^2 \end{pmatrix} \tag{16}$$

2.5 e)

And integrating over a surface (say $x = a/2$) with area A. Assume that the normal vector of the plane is in positive x direction, then one has

$$\vec{f} = f^i/A \cdot \vec{e}_i = \int dA^j T^{ij} \cdot \vec{e}_i = -2\pi\sigma^2\vec{e_x} \tag{17}$$

which shows that the force is attractive.

3 Problem 3: Stress tensor in action again

3.1 a)

Assuming that the two charges are located at $(0, L/2, 0)$ and $(0, -L/2, 0)$, then the electric field at reference point $(x, 0, 0)$ is

$$\vec{E}(x) = \frac{2q|x|}{(x^2+\frac{L^2}{4})^{3/2}}\vec{e_x} \tag{18}$$

3.2 b)

On the plane equidistant to the two charges, at reference point $(x,0,0)$,the stress tensor is

$$T^{ij} = \begin{pmatrix} -\frac{1}{8\pi}|\vec{E}|^2 & 0 & 0 \\ 0 & \frac{1}{8\pi}|\vec{E}|^2 & 0 \\ 0 & 0 & \frac{1}{8\pi}|\vec{E}|^2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{8\pi}\cdot\frac{4q^2x^2}{(x^2+\frac{L^2}{4})^3} & 0 & 0 \\ 0 & \frac{1}{8\pi}\cdot\frac{4q^2x^2}{(x^2+\frac{L^2}{4})^3} & 0 \\ 0 & 0 & \frac{1}{8\pi}\cdot\frac{4q^2x^2}{(x^2+\frac{L^2}{4})^3} \end{pmatrix} \tag{19}$$

3.3 c)

To calculate the force one integrates the stress tensor over the equidistant plane(assuming that the normal vector of the plane is in the positive y direction)

$$\vec{f} = f^i\vec{e}_i = (\int dA^jT^{ij})\vec{e}_i = \vec{e_y}\int_0^\infty \frac{1}{8\pi}\frac{4q^2r^2}{(r^2+\frac{L^2}{4})^3}2\pi r dr = \vec{e_y}\frac{q^2}{2}\int_0^\infty \frac{r^2}{(r^2+\frac{r^2}{4})^3}dr^2 = \frac{q^2}{L^2}\vec{e_y} \tag{20}$$

which shows that the force is repulsive between the two charges.

4 Problem 4: An action for Schrodinger's equation

$$S = \int dtd^3r\psi^*(i\partial_t + \frac{\hbar^2\nabla^2}{m^2} - V)\psi = \int dtd^3r(\psi^* i\partial_t\psi - \frac{\hbar^2}{m^2}(\vec{\nabla}\psi^*)\cdot(\vec{\nabla}\psi) - V\psi^*\psi) = \int dtd^3r\tilde{\mathcal{L}} \tag{21}$$

By

$$\frac{\delta\tilde{\mathcal{L}}}{\delta\psi^*} = i\partial_t\psi - V\psi \tag{22}$$

$$\frac{\delta\tilde{\mathcal{L}}}{\delta(\partial_t\psi^*)} = 0 \tag{23}$$

$$\partial_i\frac{\delta\tilde{\mathcal{L}}}{\delta(\partial_i\psi^*)} = -\frac{\hbar^2\nabla^2}{m^2}\psi \tag{24}$$

Applying Euler-Lagrange equation with respect to ψ^*one has

$$\frac{\delta\tilde{\mathcal{L}}}{\delta\psi^*} = \partial_i \frac{\delta\tilde{\mathcal{L}}}{\delta(\partial_i\psi^*)} \tag{25}$$

$$i\partial_t\psi = -\frac{\hbar^2\nabla^2}{m^2}\psi + V\psi \tag{26}$$

5 Problem 5 : Energy density of a straight wire

5.1 a)

By Ampere's law we have

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\vec{e_\phi} \tag{27}$$

5.2 b)

Energy density is simply

$$\mathcal{E} = \frac{B^2}{2\mu_0} = \frac{\mu_0}{8\pi}\frac{I^2}{r^2} \tag{28}$$

5.3 c)

Energy per unit length is

$$\begin{aligned}
\mathcal{E}_l &= \int_{\delta/2}^{R} \mathcal{E}\cdot 2\pi r dr \\
&= \frac{\mu_0 I^2}{4\pi}\int_{\delta/2}^{R}\frac{dr}{r} \\
&= \frac{\mu_0 I^2}{4\pi}\ln(\frac{2R}{\delta})
\end{aligned}$$

5.4 d)

Assuming that R is 5 m, and the thickness is 10 mm, we have

$$\begin{aligned}
\mathcal{E}_l &= \frac{\mu_0 I^2}{4\pi}\ln(\frac{2R}{\delta}) \\
&= (10^{-7})\cdot(20)^2\cdot\ln(\frac{2\cdot 5}{10\times 10^{-3}})J/m \\
&= 2.7\times 10^{-4}J/m
\end{aligned}$$