ELECTROMAGNETIC WAVES

in vacuum: $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu = 0$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \underbrace{\Box}_{\partial_\mu\partial^\mu} A^\nu - \partial^\nu \partial\cdot A = 0$$

in Lorentz gauge $\partial_\mu A^\mu = 0 \Rightarrow \underbrace{\Box}_{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2} A^\nu = 0$ (wave equation)

$$\Downarrow$$

$A_\mu = f_\mu(x \pm ct)$ is a sol. of the wave eq.

↖ any function

> E&M fields can propagate and exist far away from any currents or charges

Plane waves

$$A_\mu(x) = \mathcal{A}_\mu e^{-ik\cdot x}$$

↑ taking the real part is implicit

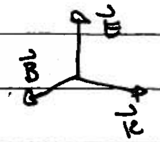
↖ complex vector

$\Box A_\mu = 0 \Rightarrow -k^2 A_\mu = 0 \Rightarrow k^2 = k_0^2 - \vec{k}^2 = 0$

$\partial_\mu A^\nu = 0 \Rightarrow -i\, k\cdot A = 0 \Rightarrow k_0 A_0 - \vec{k}\cdot\vec{A} = 0$

$\vec{E} = \vec{\mathcal{E}}\, e^{-ik\cdot x}$

$\vec{B} = \vec{\mathcal{B}}\, e^{-ik\cdot x}$



$\nabla\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E} = 0$

$\nabla\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B} = 0$

$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Rightarrow i\vec{k}\times\vec{E} + (-ik^0)\vec{B} = 0$, $\hat{k}\times\vec{E} = \vec{B}$, $|\vec{E}| = |\vec{B}|$ ($k^0 = -\omega < 0$ describes a wave moving in the $-\vec{k}$ direction)

$$\mathrm{Re}\, f(t)\ \mathrm{Re}\, g(t) = \mathrm{Re}\,|f|\, e^{-i(\omega t+\varphi)}\ \mathrm{Re}\,|g|\, e^{-i(\omega t+\theta)} \qquad \left(f(t) = f e^{i\varphi} e^{-i\omega t} \right.$$

$$= |f|\,|g| \cos(\omega t+\varphi)\cos(\omega t+\theta)$$

$$\overline{\mathrm{Re}\, f(t)\ \mathrm{Re}\, g(t)} \equiv \frac{1}{T}\int_0^T dt\ \mathrm{Re}\, f(t)\ \mathrm{Re}\, g(t) = \frac{|f|\,|g|}{4T}\int_0^T dt \left[e^{-i(\omega t+\varphi)} + e^{+i(\omega t+\varphi)} \right] \left[e^{-i(\omega t+\theta)} + e^{i(\omega t+\theta)} \right]$$

$$= \frac{|f|\,|g|}{4T}\int_0^T dt \left[e^{-i(2\omega t+\varphi+\theta)} + e^{i(2\omega t+\varphi+\theta)} + e^{-i(\varphi-\theta)} + e^{i(\varphi-\theta)} \right]$$

$$= \frac{|f|\,|g|}{2}\cos(\varphi-\theta)$$

$$\mathrm{Re}\,\frac{1}{2} f g^* = \frac{1}{2}\mathrm{Re}\,|f|\,|g|\, e^{-i(\varphi-\theta)} = \frac{|f|\,|g|}{2}\cos(\varphi-\theta)$$

$$\Downarrow$$

$$\boxed{\begin{gathered} \frac{1}{2} f g^* = \overline{fg} \quad \leftarrow \text{average over a cycle} \\ \left(\mathrm{Re}\left(\frac{1}{2} f g^*\right) = \overline{\mathrm{Re} f\ \mathrm{Re} g} \right) \end{gathered}}$$

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energy-momentum tensor for a plane wave :

energy:

$$u = \frac{\vec{E}^2+\vec{B}^2}{8\pi}, \quad \overline{U} = \frac{|\vec{E}|^2+|\vec{B}|^2}{16\pi} = \frac{|\vec{E}|^2}{8\pi}$$

Poyting vector:

$$\vec{g} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}|\vec{E}|^2\,\hat{k} = c\,u\,\hat{k}$$

Maxwell's stress tensor:

~~$T^{ij} = \frac{1}{4\pi}\left(F^{i\ell}F_\ell{}^{j} - \frac{1}{4} g^{ij} F_{mn}F^{mn}\right)$~~

$$\overleftrightarrow{T}_M = \frac{1}{4\pi}\left[\overleftarrow{E}\vec{E} + \overleftarrow{B}\vec{B} - \frac{1}{2}(E^2+B^2) \right]$$

~~$T^{zz} = $ [illegible] $F_{ij}B^{ij}$~~

$$= u\left[\hat{x}\hat{x} + \hat{y}\hat{y} - \frac{1}{2} 2 \right]$$

$$= -u\,\hat{z}\hat{z}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -u \end{pmatrix}$$

EXAMPLE: Radiation pressure

$g \simeq 1500\ \mathrm{W/m^2}$ (sunlight)

$F \simeq \hat{z}\cdot\overleftrightarrow{T}\cdot\hat{z} \simeq u = \frac{g}{c} \sim 1500\ \frac{\mathrm{W}}{\mathrm{m^2}}\ \frac{\mathrm{s}}{3\times10^8\mathrm{m}} \simeq 5\times10^{-6}\ \underbrace{\frac{\mathrm{J}}{\mathrm{s}}\ \frac{\mathrm{s}}{\mathrm{m^3}}}$

$\simeq 5\ \mu\mathrm{Pa}$ (but rad. pressure is dominant on giant stars) $\frac{\mathrm{N\,m}}{\mathrm{m^3}} = \mathrm{N/m^2} = \mathrm{Pa}$

$u = 5\times10^{-6}\ \mathrm{J/m^3}$

$|E| \simeq \sqrt{4\pi\times5\times10^{-6}\ \mathrm{J/m^3}} \simeq \sqrt{5\times10^{-5}\ \mathrm{J/m^3}} \simeq 10^{-2}\ \underbrace{\frac{\mathrm{J^{1/2}}}{\mathrm{m^{3/2}}}}$

$\sqrt{\frac{\mathrm{J}}{\mathrm{m^3}}} = \sqrt{\mathrm{Kg}\frac{\mathrm{m^2}}{\mathrm{s^2}}\frac{1}{\mathrm{m^3}}} = \sqrt{\frac{10^3\,\mathrm{g}}{\mathrm{s^2}\,10^2\mathrm{cm}}}$

$= \sqrt{10}\ \sqrt{\frac{\mathrm{g}}{\mathrm{cm\,s^2}}}$

$\simeq 10^{-2}\sqrt{10}\times3\times10^4\ \mathrm{V/m}$ $= \sqrt{10}$ statvolt/cm

$\simeq 1000\ \mathrm{V/m}$

↑ verify!

(check $(E)^2 \sim \frac{Q^2}{L^4} \sim \frac{Q^2}{L^2}\frac{1}{L^2}$

$\sim$ force $\frac{1}{L^2}$

$\sim \mathrm{g}\,\frac{\mathrm{cm}}{\mathrm{s^2}}\,\frac{1}{\mathrm{cm^2}}$

$\sim \frac{\mathrm{g}}{\mathrm{cm\,s^2}}$)

Polarization

$$\vec{E} = \vec{\varepsilon}\ e^{-ik\cdot x}$$

complex vector

$$\vec{\varepsilon}^{\,2} = |\vec{\varepsilon}|^2\ e^{-2i\alpha}\ , \quad \vec{\varepsilon} = \vec{b}\ e^{-i\alpha} \Rightarrow \vec{b}^{\,2} = \text{real}$$

define $\vec{b}$ = complex vector

$$\vec{b} = \vec{b}_1 + i\vec{b}_2\ , \quad \vec{b}^{\,2} = \text{real} \Rightarrow \vec{b}_1^{\,2} - \vec{b}_2^{\,2} + 2i\,\vec{b}_1\cdot\vec{b}_2 \Rightarrow \vec{b}_1\cdot\vec{b}_2 = 0$$

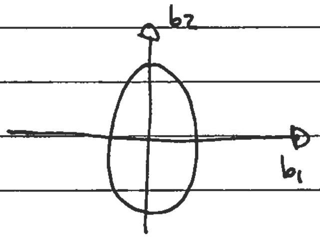
real

$$|b_1| = |b_2|$$

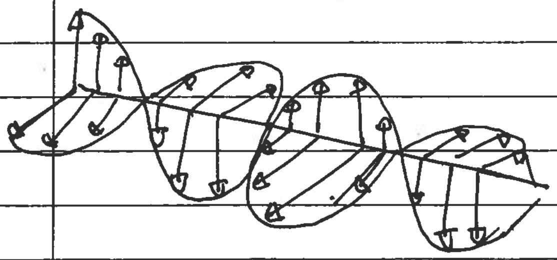
$$\Downarrow$$

$$\vec{E} = \mathrm{Re}(\vec{b}_1 + i\vec{b}_2)\ e^{-i(kx+\alpha)}$$

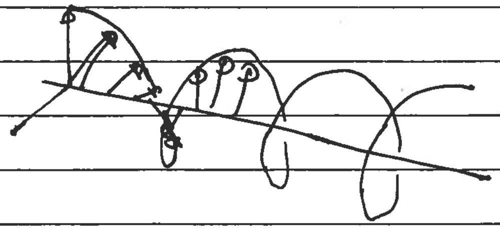
$$= \vec{b}_1\ \cos(k\cdot x+\alpha) + \vec{b}_2\ \sin(k\cdot x+\alpha)$$



linearly polarized



circularly polarized



Waves in matter

$$\nabla\cdot D = 4\pi\rho \qquad \nabla\cdot B = 0$$

$$\nabla\times E = -\frac{1}{c}\frac{\partial}{\partial t}B \qquad \nabla\times H = \frac{1}{c}\frac{\partial}{\partial t}D + \frac{4\pi}{c}J$$

$$D = \epsilon E, \quad B = \mu H$$

} valid for linear, isotropic media,
for low frequencies (time-local),
...

typically $\epsilon - 1 \gg \underbrace{|\mu - 1|}_{10^{-5} \text{ for para, diamagnets}}$

velocity: $\underbrace{\nabla\times\nabla\times E}_{-\nabla^2 E + \nabla\underbrace{\nabla\cdot E}_{=0}} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times B = -\frac{1}{c}\frac{\partial}{\partial t}\mu\epsilon\frac{\partial}{\partial t}E = -\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}E$

⇓

$$\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} - \nabla^2\vec{E} = 0$$

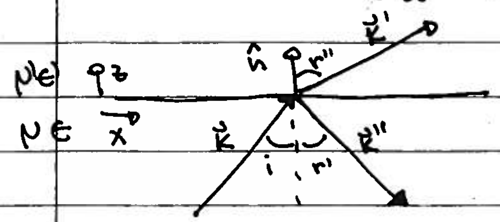
$$\frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} - \nabla^2\vec{B} = 0$$

phase velocity $= \frac{c}{\sqrt{\mu\epsilon}} << c \quad (\epsilon > 1)$

$n = \sqrt{\mu\epsilon}$ = index of refraction

$\vec{E} = \vec{\mathcal{E}} e^{-ik\cdot x} \Rightarrow n^2 k_0^2 - \vec{k}^2 = 0, \quad |k_0| = \frac{|k|}{n}$

$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B} \Rightarrow i\vec{k}\times\vec{E} - ik^0\vec{B} = 0, \quad \vec{k}\times\vec{E} = \frac{|k|}{n}\vec{B}$



INCIDENT: $\vec{E} = \vec{\mathcal{E}} e^{-ikx}, \quad \vec{B} = \vec{\mathcal{B}} e^{-ikx} = \sqrt{\mu\epsilon}\,\hat{k}\times\vec{E}$

REFRACTED: $\vec{E}' = \vec{\mathcal{E}}' e^{-ik'\cdot x}, \quad \vec{B}' = \vec{\mathcal{B}}' e^{-ik'\cdot x} = \sqrt{\mu'\epsilon'}\,\hat{k}'\times\vec{E}'$

REFLECTED: $\vec{E}'' = \vec{\mathcal{E}}'' e^{-ik''\cdot x}, \quad \vec{B}'' = \vec{\mathcal{B}}'' e^{-ik''\cdot x} = \sqrt{\mu\epsilon}\,\hat{k}''\times\vec{E}''$

$|\vec{k}| = |\vec{k}''| \equiv k = \frac{\omega}{c}\sqrt{\mu\epsilon}$ (

$|\vec{k}'| = k' = \frac{\omega}{c}\sqrt{\mu'\epsilon'}$

$(\vec{k}\cdot\vec{r})\big|_{z=0} = (\vec{k}'\cdot\vec{r})\big|_{z=0} = (\vec{k}''\cdot r)\big|_{z=0}$ (phases matching @ boundary) y-component: $k_y = k_y' = k_y'' = 0$ (choose $k_y = 0$, the others then vanish)

x-component: $$\underbrace{\underbrace{k \sin i = \underset{k}{\underline{k''}} \sin r''}_{i=r} = k' \sin r}$$.

$$\frac{\sin i}{\sin r} = \frac{k'}{k} = \sqrt{\frac{\mu'\epsilon'}{\mu\epsilon}} = \frac{n'}{n}$$ (Snell's law)

The relations above depend only on general wave properties and are true for any wave like, i.e., sound.

b.c.'s for E, B:

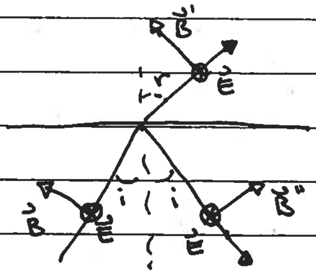
$D_\perp = D_\perp$: $\epsilon(\vec{\mathcal{E}} + \vec{\mathcal{E}}'')\cdot\hat{n} = \epsilon'\vec{\mathcal{E}}'\cdot\hat{n}$ (i)

$E_\parallel = E_\parallel$: $(\vec{\mathcal{E}} + \vec{\mathcal{E}}'')\times\hat{n} = \vec{\mathcal{E}}'\times\hat{n}$ (ii)

$B_\perp = B_\perp$: $(\vec{k}\times\vec{\mathcal{E}} + \vec{k}''\times\vec{\mathcal{E}}'')\cdot\hat{n} = \hat{k}'\times\vec{\mathcal{E}}\cdot\hat{n}$ (iii)

$H_\parallel = H_\parallel$: $\frac{1}{\mu}(\vec{k}\times\vec{\mathcal{E}} + \vec{k}''\times\vec{\mathcal{E}}')\times\hat{n} = \frac{1}{\mu'}(\hat{k}'\times\vec{\mathcal{E}})\times\hat{n}$ (iv)

$\vec{\mathcal{E}} \perp \hat{k}, \vec{k}''$



ii) $\mathcal{E} + \mathcal{E}'' = \mathcal{E}'$

iv) $\sqrt{\frac{\epsilon}{\mu}}(\mathcal{E} - \mathcal{E}'')\cos i = \sqrt{\frac{\epsilon'}{\mu'}}\,\mathcal{E}'\cos r$

~~$\sqrt{\frac{\epsilon}{\mu}}\,\mathcal{E}\cos i = \sqrt{\frac{\epsilon'}{\mu'}}\,\mathcal{E}'\cos r$~~

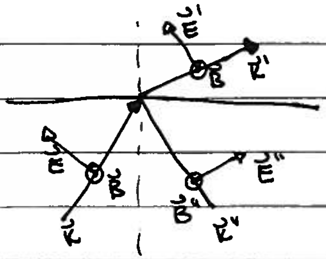
$$\sqrt{\frac{\epsilon}{\mu}}(2\mathcal{E} - \mathcal{E}')\cos i = \sqrt{\frac{\epsilon'}{\mu'}}\,\mathcal{E}'\cos r$$

$$\mathcal{E}'\left[\sqrt{\frac{\epsilon'}{\mu'}}\cos r + \sqrt{\frac{\epsilon}{\mu}}\cos i\right] = 2\sqrt{\frac{\epsilon}{\mu}}\,\mathcal{E}$$

$$\mathcal{E}' = 2\sqrt{\frac{\epsilon}{\mu}}\,\frac{\mathcal{E}\cos i}{\sqrt{\frac{\epsilon'}{\mu'}}\cos r + \sqrt{\frac{\epsilon}{\mu}}\cos i}$$

$$\frac{\mathcal{E}'}{\mathcal{E}} = \frac{2n\cos i}{n\cos i + \sqrt{\frac{\epsilon'\mu^2}{\mu'}}\underbrace{\sqrt{1 - \frac{n^2}{n'^2}\sin^2 i}}_{\cos r}} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

$\vec{E} \parallel \vec{k},\vec{k}''$



$$\text{iii)}\quad (\mathcal{E}-\mathcal{E}'')\cos i = \mathcal{E}'\cos r \quad \Rightarrow \quad \mathcal{E}'' = \mathcal{E} - \mathcal{E}'\frac{\cos r}{\cos i}$$

$$\text{iv)}\quad \sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E}+\mathcal{E}''\right)\cancel{\sin i} = \sqrt{\frac{\epsilon'}{\mu'}}\,\mathcal{E}'\,\cancel{\sin r}$$

$$\sqrt{\frac{\epsilon}{\mu}}\left(2\mathcal{E} - \mathcal{E}'\frac{\cos r}{\cos i}\right)\cancel{\sin i} = \sqrt{\frac{\epsilon'}{\mu'}}\,\mathcal{E}'\,\cancel{\sin r}$$

$$\mathcal{E}'\left[\sqrt{\frac{\epsilon'}{\mu'}}\underbrace{\cancel{\sin r}}_{\cancel{\frac{n}{n'}\sin i}} + \sqrt{\frac{\epsilon}{\mu}}\underbrace{\frac{\cos r\,\cancel{\sin i}}{\cos i}}_{\frac{\cancel{\sin i}}{\cos i}\sqrt{1-\frac{n^2}{n'^2}\sin^2 i}}\right] = 2\sqrt{\frac{\epsilon}{\mu}}\,\cancel{\sin i}\,\mathcal{E}$$

$$\frac{\mathcal{E}'}{\mathcal{E}} = 2\sqrt{\frac{\epsilon}{\mu}}\;\frac{1}{\frac{n'}{\mu'} + \frac{1}{\mu}\frac{n}{n'\cos i}\sqrt{n'^2-n^2\sin^2 i}}$$

$$= \frac{2n\cos i}{n'\frac{\mu}{\mu'}\cos i + \frac{n}{n'}\sqrt{n'^2-n^2\sin^2 i}}$$

$$= \frac{2nn'\cos i}{n'^2\frac{\mu}{\mu'}\cos i + n\sqrt{n'^2-n^2\sin^2 i}}$$

- for $\vec{E} \parallel$ to $\vec{k},\vec{k}'$, $\mu=\mu'$, ~~$n'^2\cos i + n\sqrt{n'^2-n^2\sin^2 i}=0$~~
~~$\Rightarrow n'^4\cos^2 i = n^2(n'^2-n^2\sin^2 i)$~~
~~$n^2 n'^2$~~

$$\frac{\mathcal{E}''}{\mathcal{E}} = \frac{\mathcal{E}-\mathcal{E}'\cos r/\cos i}{\mathcal{E}} = 1 - \frac{\mathcal{E}'}{\mathcal{E}}\frac{\sqrt{1-n^2/n'^2\sin^2 i}}{\cos i}$$

$$= 1 - \frac{\sqrt{n'^2-n^2\sin^2 i}}{\cancel{n'}\cancel{\cos i}}\;\frac{2n\cancel{n'}\cancel{\cos i}}{n'^2\cos i + n\sqrt{n'^2-n^2\sin^2 i}}$$

$$= \frac{n'^2\cos i + n\sqrt{n'^2-n^2\sin^2 i} \overset{2n}{-} \sqrt{n'^2-n^2\sin^2 i}}{n'^2\cos i + n\sqrt{n'^2-n^2\sin^2 i}} = 0$$

(Brewster's angle).

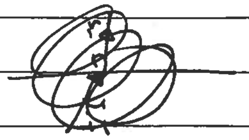
$$n'^2\cos i = +n\sqrt{n'^2-n^2\sin^2 i} \Rightarrow \operatorname{tg} i = \frac{n'}{n}$$

~~$\left(\frac{n'^2}{n}\right)^2\cos^2 i = n'^2 - n^2\sin^2 i$~~

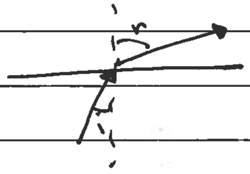
There's a tendency for the reflected wave to be polarized perpendicular to $\vec{k}, \vec{k}''$ (Ray-Ban's !)

- Total internal reflection

If $n > n'$, $\frac{\sin i}{\sin r} = \frac{n'}{n} \cancel{>} < 1 \Rightarrow i \cancel{>} r < \pi/2$



$\sin i > n'/n \Rightarrow$ no refraction



In the $\sin i > n'/n$ case, $\sin r = \frac{n}{n'} \sin i > 1$

$$\cos r = \sqrt{1 - \frac{n^2}{n'^2}\sin^2 i} = i\sqrt{\frac{n^2}{n'^2}\sin^2 i - 1}$$

$$\Rightarrow e^{i\vec{k}'\cdot\vec{r}} = e^{ik'(x\sin r + z\cos r)} = e^{ik'x\sin r}\ \underset{\text{exponential decay in } z}{e^{-\sqrt{\frac{n^2}{n'^2}\sin^2 i - 1}\, z}}$$

~~[illegible]~~

"light pipes" and $\underset{\sim\sim}{\text{"fiber optics"}}$

in between internal reflection ($\lambda \ll L$)
and waveguide ($\lambda \sim L$)

Frequency-dependent

~~A model for~~ dielectrics $D = \epsilon E$ is too simplistic for many applications

$$D(t,\vec{r}) = \int_{-\infty}^{\infty} d\tau \; \epsilon(\tau) \; E(t-\tau,\vec{r})$$ (non-locality in time, memory)

drop $\vec{r}$'s :

$$\left.\begin{matrix} D(t) \\ E(t) \\ \epsilon(t) \end{matrix}\right\} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left\{\begin{matrix} D(\omega) \\ E(\omega) \\ \epsilon(\omega) \end{matrix}\right.$$

$$\left.\begin{matrix} D(\omega) \\ E(\omega) \\ \epsilon(\omega) \end{matrix}\right\} = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \left\{\begin{matrix} D(t) \\ E(t) \\ \epsilon(t) \end{matrix}\right.$$

$$D(t) = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \epsilon(\omega) \int_{-\infty}^{\infty} \frac{d\omega'}{(2\pi)} e^{-i\omega'(t-\tau)} E(\omega')$$

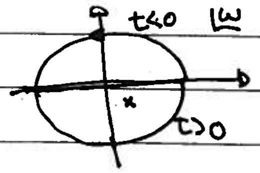
$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \underbrace{\epsilon(\omega) E(\omega)}_{D(\omega)}$$

$$D(\omega) = \epsilon(\omega) \, E(\omega)$$

causality : $\epsilon(t) = 0$ for $t < 0$ ⇨ $\epsilon(\omega) = \int_0^{\infty} d\tau \; e^{i\omega t} \epsilon(t)$ analytically in upper plane

$$\epsilon(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \epsilon(\omega)$$

$$= \begin{cases} 0, & t<0 \\ \neq 0, & t>0 \end{cases}$$



if $\epsilon(t)$ doesn't grow exponentially

Kramers-Krönig relation $$\underset{\substack{\uparrow \\ \text{upper plane}}}{\varepsilon(\omega)} - 1 = \frac{1}{2\pi i}\oint d\omega' \, \frac{\varepsilon(\omega')-1}{\omega'-\omega}$$

goes to zero as $R \to \infty$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \, \frac{\varepsilon(\omega')-1}{\omega'-\omega-i\eta}$$

positive infinitesimal

$$\frac{1}{\omega'-\omega-i\eta} = \frac{\omega'-\omega+i\eta}{(\omega'-\omega)^2+\eta^2} = \underbrace{\frac{\omega'-\omega}{(\omega'-\omega)^2+\eta^2}}_{P\left(\frac{1}{\omega'-\omega}\right)} + i\underbrace{\frac{\eta}{(\omega'-\omega)^2+\eta^2}}_{i\pi\delta(\omega'-\omega)}$$

$$\varepsilon(\omega)-1 = \frac{1}{2\pi i} P\int_{-\infty}^{\infty} d\omega' \, \frac{\varepsilon(\omega')-1}{\omega'-\omega} + \frac{1}{2}\left(\varepsilon(\omega)-1\right)$$

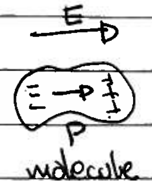
$$\frac{\varepsilon(\omega)-1}{2} = P\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi i}\,\frac{\varepsilon(\omega')-1}{\omega'-\omega}$$

$$\boxed{\begin{aligned} \operatorname{Re}\varepsilon(\omega)-1 &= P\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\,\frac{\operatorname{Im}\varepsilon(\omega')}{\omega'-\omega} \\ \operatorname{Im}\varepsilon(\omega) &= -P\int_{-\infty}^{\infty}\frac{d\omega'}{\pi}\,\frac{\operatorname{Re}\varepsilon(\omega')-1}{\omega'-\omega} \end{aligned}}$$

Re ε(ω) determines Im ε(ω) and vice-versa. Sometimes it's easy to measure Im ε(ω) but not Re ε(ω).

physical interpretation is coming up!

A model for dielectrics



$$m\left[\ddot{x} + \underset{\substack{\uparrow \\ \text{dissipation}}}{\gamma}\dot{x} + \underset{\substack{\uparrow \\ \text{frequency of natural oscillation}}}{\omega_0^2}x\right] = eE$$

$$\Downarrow$$

$$-m\omega^2 x(\omega) + i\omega\gamma x(\omega) + m\omega_0^2 x(\omega) = eE(\omega)$$

$$\Downarrow$$

$$x(\omega) = \frac{e E(\omega)}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$P(\omega) = \underset{\substack{\uparrow \\ \text{density of molecules}}}{N} e\, x(\omega) = \underbrace{\sum_i \frac{e^2 N}{m} \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}}_{\chi_e(\omega)} E(\omega$$

$$\boxed{\epsilon(\omega) = 1 + 4\pi\chi_e(\omega) = 1 + \frac{4\pi e^2 N}{m} \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma}}$$

Interesting limits

$$\omega \gg \omega_i, \gamma \Rightarrow \epsilon(\omega) = 1 - \underbrace{\frac{4\pi e^2 N}{m}}_{\equiv \omega_p^2 = \text{plasma frequency}} \frac{1}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\underbrace{n^2 k_0^2}_{= \epsilon(\omega)\frac{\omega^2}{c^2}} - \vec{k}^2 = 0 \Rightarrow \left(1 - \frac{\omega_p^2}{\omega^2}\right)\frac{\omega^2}{c^2} - k^2 = 0$$

$$\omega^2 = \omega_p^2 + c^2 k^2$$

in QM, this is like a mass for the photon

⑫

similar thing happens as plasmas $(\omega_0 \simeq \gamma \simeq 0) \Rightarrow \epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$
even for $\omega < \omega_p$

$$\omega < \omega_p, \quad \epsilon(\omega) < 0 \Rightarrow K = \sqrt{\frac{\omega^2}{c^2} n^2} = i\,|n|\frac{\omega}{c}$$

$$\text{plane wave} \sim e^{-ikx} = e^{-i\omega t}\, \underbrace{e^{i\, i|n|\omega/c\, x}}_{e^{-\frac{\omega|n|}{c}x}}$$

$$\text{penetration length} = \frac{c}{\omega|n|} \xrightarrow[\omega \to 0]{} \frac{c}{\omega}\,\frac{\omega}{\omega_p} = \frac{c}{\omega_p}$$

Conductors

all $\omega_i \neq 0 \Rightarrow$ restorative force, no electrons to conduct

one $\omega_i = 0 \Rightarrow \epsilon(\omega) = \underbrace{1 - \frac{4\pi e^2 N}{m} \sum_{i \neq 0} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma}}_{\equiv \epsilon_0} + i\frac{4\pi e^2 N}{m\gamma}\frac{1}{\omega}$

ohmic current, real $\epsilon_0(\omega)$

$$\nabla \times H = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial D}{\partial t} = \frac{4\pi\sigma}{c} E - \frac{i\omega}{c}\epsilon_0 E = -\frac{i\omega}{c}\left[\epsilon_0 + i\frac{4\pi\sigma}{\omega}\right.$$

$$\nabla \times H = \frac{1}{c}\frac{\partial}{\partial t} D = \frac{1}{c}(-i\omega)\epsilon E$$

no current, complex $\epsilon(\omega)$

$$\epsilon = \epsilon_0 + i\frac{4\pi\sigma}{\omega}$$

in our model $\Rightarrow \sigma = \frac{e^2 N}{m\gamma}$ (Drude's model result

The imaginary part of $\epsilon(\omega)$ can be thougth as a conductivity;

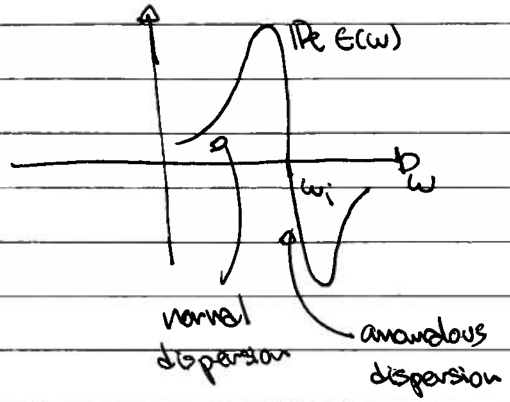
in either case the $\bar{e}$ do a harmonic motion

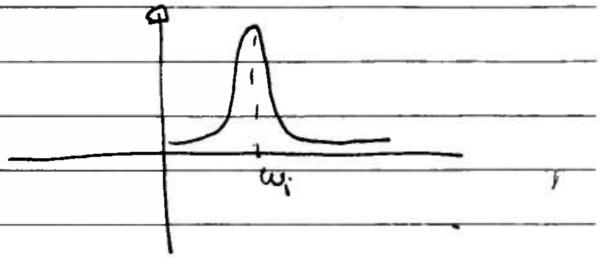
ϵ real $\quad \overleftarrow{E} \quad \overleftarrow{P} \quad p(\omega) \sim x(\omega) \sim E(\omega)$

ϵ imaginary $\quad \overleftarrow{E} \quad \overleftarrow{x} \; \overrightarrow{P} \quad p(\omega) \sim x(\omega) \sim \frac{i}{\omega}\underbrace{x(\omega)}_{\sim J} \sim \frac{i}{\omega}E(\omega)$

around a resonance

$$\omega \simeq \omega_i \Rightarrow \epsilon(\omega) = 1 + \frac{4\pi e^2 N}{m} \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma}$$





Waves in conductors

little dependence on ω

$$\epsilon(\omega) = \underbrace{\epsilon_R(\omega)}_{\text{from now on simply } \epsilon(\omega)} + i\frac{4\pi\sigma(\omega)}{\omega}$$

plane wave ~ $e^{-ik\cdot x} = e^{-i\omega t} e^{ik\cdot r}$, $n^2\frac{\omega^2}{c^2} = k^2$

$$\Rightarrow k^2 = \mu\epsilon\frac{\omega^2}{c^2}\left[1 + i\frac{4\pi\sigma}{\epsilon\omega}\right]$$

$$\Rightarrow k = \sqrt{\mu\epsilon}\,\frac{\omega}{c}\left[\sqrt{\frac{1}{2}\sqrt{1+\left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2}+1} + i\sqrt{\frac{1}{2}\sqrt{1+\left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2}-1}\right]$$

(pick branch to reproduce σ=0 result)

$$\equiv \beta + i\frac{\alpha}{2}$$

$$\begin{cases}\vec{E} \\ \vec{B}\end{cases} = \begin{cases}\vec{\mathcal{E}} \\ \vec{\mathcal{B}}\end{cases} e^{-ikx} = \begin{cases}\vec{\mathcal{E}} \\ \vec{\mathcal{B}}\end{cases} e^{-i\omega t + i\vec{\beta}\cdot\vec{r}}\, e^{-\frac{\vec{\alpha}}{2}\cdot\vec{r}}$$

$$\nabla\times E + \frac{1}{c}\frac{\partial}{\partial t}B = 0 \Rightarrow i\vec{k}\times\vec{\mathcal{E}} - \frac{i\omega}{c}\vec{\mathcal{B}} \Rightarrow \vec{\mathcal{B}} = \frac{\vec{k}}{\omega}\times\vec{\mathcal{E}} = \frac{c}{\omega}\left(\beta + \frac{i\alpha}{2}\right)\hat{k}\times\vec{\mathcal{E}}$$

$\vec{B}$ is out of phase w/ $\vec{E}$

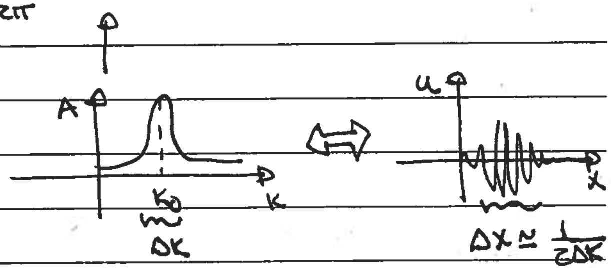
$|\vec{B}| \neq |\vec{E}|$

skin depth
or
penetration length
$$\delta = \frac{2}{\alpha} \underset{\sigma \gg \omega}{\simeq} \sqrt{\frac{2\epsilon\omega}{4\pi\sigma}} \frac{c}{\omega} \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{2\pi\sigma\mu\omega}}$$

WAVE PACKETS AND GROUP VELOCITY

Take $u(x,t)$ to be any wave $\qquad U(x,t) \sim e^{-i\omega(k)t + ikx}$ (plane wave
(a component of $\vec{E}$ or $\vec{B}$)

wave packet: $\quad u(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{-i\omega(k)t + ikx}$



$$u(x,t) \simeq \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{-i\left[\omega(k_0) + (k-k_0)\omega'(k_0) + \cdots\right]t + ikx}$$

$$= e^{-i\omega(k_0)t} \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{-i(\omega'(k_0)t - x)k + \cdots}$$

$$\simeq e^{-i\omega(k_0)t}\, u(x - \omega'(k_0)t, 0)$$

$|u(x,t)|$ moves with velocity $\left.\frac{d\omega}{dk}\right|_{k=k_0} \equiv V_g$

For E&M waves $\quad \omega = \frac{ck}{n(k)} \Rightarrow V_g = \frac{d\omega}{dk} = \frac{c}{n} - \frac{ck}{n^2} \underbrace{\frac{dn}{dk}}_{\frac{dn}{d\omega}\frac{d\omega}{dk}}$

$$V_g = \frac{c}{n}\left[1 - \frac{k}{n} V_g \frac{dn}{d\omega}\right]$$

$$V_g\left[1 + \frac{ck}{n^2}\frac{dn}{d\omega}\right] = \frac{c}{n}$$

$$V_g = \frac{c}{n} \frac{1}{1 + \frac{ck}{n^2}\frac{dn}{d\omega}} = \frac{c}{n + \frac{\text{[illegible]}}{\omega}\frac{dn}{d\omega}}$$

GEOMETRICAL OPTICS

- one single frequency. u stands for any component of $\vec{E}$ or $\vec{B}$

$$\frac{n^2\omega^2}{c^2}u + \nabla^2 u = 0$$

Typical distance over which n changes

Let's look for solutions with short wavelength. In that limit ($\lambda \ll \frac{n}{|\nabla n|}$) there's a well defined concept of "trajectory of light rays".

e^{ikx}

$k\lambda = 2\pi$

$\frac{\lambda}{2\pi} = \frac{1}{k}$

$\omega = ck$

$$\text{ansatz}: \quad u(\vec{r}) = \underset{\substack{\uparrow\\ \text{changes slowly, like } n}}{A(\vec{r})}\; \underbrace{e^{i S/\bar\lambda_0}}_{\text{changes fast}} \qquad \bar\lambda_0 \equiv \frac{\lambda_0}{2\pi} = \frac{1}{k_0} = \frac{c}{\omega} = \frac{1}{k_0}$$

$P(\vec{r})$

$$\bar\lambda \equiv \frac{c}{n\omega} = \frac{1}{k}$$

$$\nabla u = \left(\nabla A + \frac{i}{\bar\lambda}\nabla S\right) \quad e^{iS/\bar\lambda_0}$$

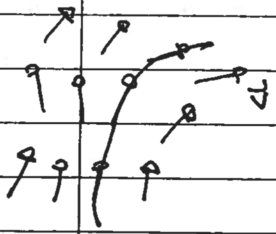
$$\frac{n^2\omega^2}{c^2}u + \nabla^2 u = \left(\nabla^2 A + \frac{i}{\bar\lambda_0}\nabla^2 S + \frac{i}{\bar\lambda_0}\nabla A\cdot\nabla S - \frac{A(\nabla S)^2}{\bar\lambda_0^2}\right)e^{iS/\bar\lambda} + \frac{1}{\bar\lambda^2}Ae^{iS/\bar\lambda}$$

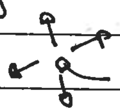
$$\Downarrow \; \bar\lambda, \bar\lambda_0 \to 0$$

$$\boxed{\begin{aligned}(\nabla S)^2 &= n^2\\ A\nabla^2 S + \nabla S\cdot\nabla A &= 0\end{aligned}}$$

$$\frac{|\nabla S|}{\bar\lambda_0} = \frac{2\pi}{\bar\lambda}$$

"local" wavelength





$$\nabla\cdot\nabla S > 0 \Rightarrow \frac{\nabla A}{A}\cdot\nabla S < 0$$

(amplitude decreases along ∇S)

$$\vec{v} = \frac{1}{n}\nabla S, \quad \vec{v}^2 = 1$$

$$\underbrace{\nabla\times(n\vec{v})}_{\nabla n\times\vec{v} + n\nabla\times\vec{v}} = \nabla\times\nabla S = 0$$

$$\nabla n\times\vec{v} + n\nabla\times\vec{v} \Rightarrow \nabla\times\vec{v} = -\frac{1}{n}\nabla n\times\vec{v}$$

$$\frac{d\vec{x}}{ds} = \vec{v}, \qquad \frac{dv^i}{ds} = v^j\partial_j v^i$$

ds: length along the curve

$$\|$$

$$\left[(\nabla\times\vec{v})\times\vec{v}\right]_i = \underbrace{\overbrace{\epsilon^{ikl}}^{kli}\epsilon^{kjm}}_{\delta^{lj}\delta^{im} - \delta^{lm}\delta^{ij}}\partial_j v_m v^l = v^j v_j v_i - \overset{0}{\cancel{\partial_i \frac{v^2}{2}}}$$

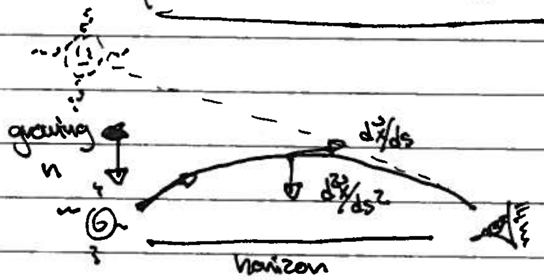
$$\frac{d^2x^i}{ds^2} = ((\nabla\times\vec{v})\times\vec{v})^i = -\frac{1}{n}((\nabla n\times\vec{v})\times\vec{v})^i = -\frac{1}{n}\,\epsilon^{\overset{k\ell j}{\widetilde{\ell j k}}}\,\partial_j n\, v_k\, v^m\, \epsilon^{i}{}_{\ell m}$$

$$= -\frac{1}{n}\left(\delta^{ki}\delta^{jm} - \delta^{km}\delta^{ji}\right)\partial_j n\, v_k v_m$$

$$= -\frac{1}{n}\left(v_j \partial_j n\, v_i - \partial_i n\, v^2\right)$$

⇩

$$\boxed{n\frac{d^2\vec{x}}{ds^2} = \underbrace{\nabla n - \nabla . n\frac{d\vec{x}}{ds}\frac{d\vec{x}}{ds}}_{\text{projection of } \nabla n \text{ orthogonal to } d\vec{x}/ds=\vec{v}}}$$



Fermat principle

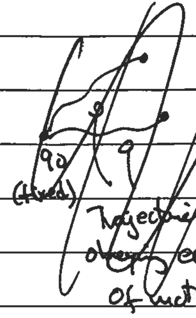
$$T = \cancel{\overbrace{\int ds}^{\text{length}} \underbrace{\frac{n(\vec{x}(s))}{c}}_{\text{velocity}}}_{dt} = \cancel{\frac{1}{c}}\int \overbrace{\underbrace{\overbrace{ds\sqrt{\left(\frac{d\vec{x}}{ds}\right)^2}}^{\text{distance}}\;\overbrace{\frac{n(\vec{x}(s))}{c}}^{1/\text{velocity}}}_{\text{"}\mathcal{L}\text{"}}}^{dt}$$

minimum of T ⇨ $\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\frac{dx^i(s)}{ds}} - \frac{\partial\mathcal{L}}{\partial x^i(s)} = \frac{d}{ds}\left[\frac{2\,dx^i/ds\; n}{2\sqrt{\;}}\right] - \sqrt{\;}\,\nabla n = 0$

choose s so $\left|\frac{d\vec{x}}{ds}\right| = 1$ ⇒ $\frac{d}{ds}\left(n\frac{d\vec{x}}{ds}\right) = \nabla n$ ⇨ $n\frac{d^2\vec{x}}{ds^2} = -\nabla n . \frac{d\vec{x}}{ds}\frac{d\vec{x}}{ds} + \nabla n$

> light travels along the fastest path

analogy w/ mechanics!



~~trajectories obeying eqs of motion~~

~~$S = \int_{t_0}^{t} dt\, L$~~

~~fixed t, vary final coordinate:~~

~~$\delta S = \frac{\partial L}{\partial \dot q^i}\delta q^i \Big|_{t_0}^{t} + \int_{t_0}^{t} dt \underbrace{\left(\frac{\partial L}{\partial q^i} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot q^i}\right)}_{=0}\delta q^i$~~

~~fixed coordinate, vary t:~~

~~$\delta S =$~~

$$n = \sqrt{2m(E - V(r))}$$

$$\nabla n = \frac{-2M\nabla V(r)}{2\sqrt{\quad}} = \frac{M\,F(r)}{\sqrt{2M(E-V)}}$$

~~$\underbrace{2M(E-V)}_{\frac{Mv^2}{2}} \underbrace{\frac{d^2\vec x}{ds^2}}_{\frac{\hat n}{R}} = 2MF \underbrace{(1 - \vec t\,\vec t)}_{\text{centripetal force}}$~~

$$\sqrt{2M(E-V)}\,\frac{d^2\vec x}{ds^2} = \frac{M\,F}{\sqrt{2M(E-V)}}(1 - \vec t\,\vec t)$$

$$\underbrace{2\cancel{M}(E-V)}_{mv^2}\ \underbrace{\frac{d^2\vec x}{ds^2}}_{\hat n/R} = \cancel{M}\underbrace{F(1 - \vec t\,\vec t)}_{\text{centripetal force}}$$

R ← radius of curvature

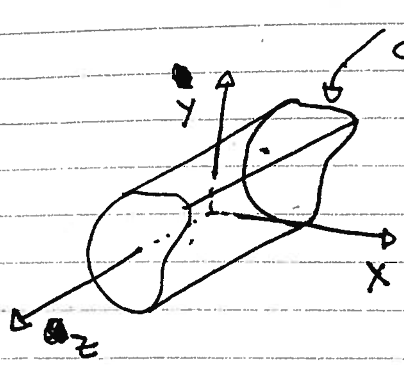
Fermat principle

$$\delta \int ds\, n = 0$$

Malpertuis principle

$$\delta \int ds \sqrt{2M(E-V)} = 0$$

WAVE GUIDES



boundary conditions

inside the conductor

$\vec{E}=0 \Rightarrow \frac{\partial \vec{B}}{\partial t}=0 \Rightarrow \vec{B}=0$ $\quad \nabla\times E+\frac{1}{c}\frac{\partial}{\partial t}B=0 \Rightarrow E_{\parallel}=0$

assuming it started as zero $\quad \nabla\cdot B=0 \Rightarrow B_{\perp}=0$

waves down the pipe:

$$\vec{E}(\vec{r},t) = \vec{\mathcal{E}}(x,y)\, e^{-i(\omega t-kz)}$$
$$\vec{B}(\vec{r},t) = \vec{\mathcal{B}}(x,y)\, e^{-i(\omega t-kz)}$$

$$\vec{\mathcal{E}} = \vec{\mathcal{E}}_{\perp} + \mathcal{E}_z\hat{z} \quad , \quad \vec{\mathcal{B}} = \vec{\mathcal{B}}_{\perp} + \mathcal{B}_z\hat{z}$$

$\nabla\times E+\frac{1}{c}\frac{\partial B}{\partial t}=0 \Rightarrow \nabla_{\perp}\times\mathcal{E}_{\perp}-\frac{i\omega}{c}\mathcal{B}_z=0$ $\quad \nabla\times B-\frac{1}{c}\frac{\partial E}{\partial t}=0 \Rightarrow \nabla_{\perp}\times\mathcal{B}_{\perp}+\frac{i\omega}{c}\mathcal{E}_z=0$

$$\underbrace{\frac{\partial\mathcal{E}_{\perp}}{\partial z}}_{ik\mathcal{E}_{\perp}} \underbrace{-\nabla_{\perp}\mathcal{E}_z} -\frac{i\omega}{c}\overset{\hat{z}\times}{\mathcal{B}_{\perp}}=0 \qquad \underbrace{\frac{\partial\mathcal{B}_{\perp}}{\partial z}}_{ik\mathcal{B}_{\perp}} \underbrace{-\nabla_{\perp}\mathcal{B}_z} +\frac{i\omega}{c}\hat{z}\times\mathcal{E}_{\perp}=0$$

$\nabla\cdot E=0 \Rightarrow \nabla_{\perp}\cdot\mathcal{E}_{\perp}+\underbrace{\frac{\partial\mathcal{E}_z}{\partial z}}_{ik\mathcal{E}_z}=0$ $\quad \nabla\cdot B=0 \Rightarrow \nabla_{\perp}\cdot\mathcal{B}_{\perp}+\underbrace{\frac{\partial\mathcal{B}_z}{\partial z}}_{ik\mathcal{B}_z}=0$

We can determine $\mathcal{E}_{\perp}$ and $\mathcal{B}_{\perp}$ if $\mathcal{E}_z$, $\mathcal{B}_z$ are known:

$$\mathcal{E}_{\perp} = -\frac{i}{k}\left[\nabla_{\perp}\mathcal{E}_z+\frac{i\omega}{c}\hat{z}\times\mathcal{B}_{\perp}\right] = -\frac{i}{k}\left[\nabla_{\perp}\mathcal{E}_z+\frac{i\omega}{c}\hat{z}\times\left(-\frac{i}{k}\nabla_{\perp}\mathcal{B}_z-\frac{\omega}{kc}\hat{z}\times\mathcal{E}_{\perp}\right)\right]$$

$$= -\frac{i}{k}\left[\nabla_{\perp}\mathcal{E}_z+\frac{\omega}{kc}\hat{z}\times\nabla_{\perp}\mathcal{B}_z+\frac{i\omega^2}{kc^2}\mathcal{E}_{\perp}\right]$$

$$\Rightarrow \mathcal{E}_{\perp}\left(1-\frac{\omega^2}{k^2c^2}\right) = -\frac{i}{k}\left[\nabla_{\perp}\mathcal{E}_z+\frac{\omega}{kc}\hat{z}\times\nabla_{\perp}\mathcal{B}_z\right]$$

$$\text{or} \quad \mathcal{E}_{\perp} = \underbrace{\frac{i}{k}\,\frac{1}{\frac{\omega^2}{k^2c^2}-1}}_{\frac{ik}{(\omega/c)^2-k^2}}\left[\nabla_{\perp}\mathcal{E}_z+\frac{\omega}{kc}\hat{z}\times\nabla_{\perp}\mathcal{B}_z\right]$$

$$\mathcal{E}_{\perp} = \frac{i}{(\omega/c)^2-k^2}\left[k\nabla_{\perp}\mathcal{E}_z+\frac{\omega}{c}\hat{z}\times\nabla_{\perp}\mathcal{B}_z\right]$$

similarly ($\mathcal{E} \leftrightarrow \mathcal{B}$, $\omega \leftrightarrow -\omega$) : $\mathcal{B}_\perp = \frac{i}{(\omega/c)^2 - k^2}\left[k \nabla_\perp \mathcal{B}_z - \frac{\omega}{c}\hat{z} \times \nabla_\perp \mathcal{E}_z\right]$

$\mathcal{E}_z$, $\mathcal{B}_z$ are determined by the 2D wave equation:

$$\left[\nabla_\perp^2 - k^2 + \left(\frac{\omega}{c}\right)^2\right]\begin{cases}\mathcal{E}_z(x,y)\\ \mathcal{B}_z(x,y)\end{cases} = 0$$

↖ 2D problem

general solution is a combination of TE, TM and TEM waves

TE : $\mathcal{E}_z = 0$

TM : $\mathcal{B}_z = 0$

TEM : $\mathcal{E}_z = \mathcal{B}_z = 0$ (doesn't exist in a hollow pipe. The 2D problem is

$$\underbrace{\begin{matrix}\nabla_\perp \times \mathcal{E}_\perp = 0\\ \nabla_\perp \cdot \mathcal{E}_\perp = 0\end{matrix}}_{\text{electrostatic}} \qquad \underbrace{\begin{matrix}\nabla_\perp \times \mathcal{B}_\perp = 0\\ \nabla_\perp \cdot \mathcal{B}_\perp = 0\end{matrix}}_{\text{magnetostatics}}$$

$\mathcal{E}_\perp = -\nabla\phi$, $\nabla^2\phi = 0$

$E_\parallel$ /conductor (boundary) $= 0$ $\Rightarrow$ $\phi(\text{boundary}) = 0$

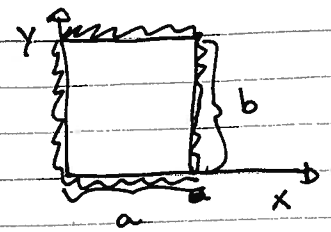
$\Rightarrow$ $\phi = 0$

$\mathcal{B}_\perp = -\nabla\phi_m$, $\nabla^2\phi_m = 0$

$\mathcal{B}_\perp(\text{boundary}) = 0$ $\Rightarrow$ $\frac{\partial\phi_m}{\partial n}(\text{boundary}) = 0$

$\Rightarrow$ $\phi_m = 0$

EXAMPLE: rectangular waveguide



$$\left[\nabla_\perp^2 - k^2 + \left(\frac{\omega}{c}\right)^2\right]\left\{\begin{matrix} E_z \\ B_z \end{matrix}\right. = 0$$

TM modes: $B_z = 0$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \left(\frac{\omega}{c}\right)^2\right] E_z(x,y) = 0$$

$$E_z(\text{boundary}) = 0$$

$$\Downarrow$$

called TM_{nm} mode

$$E_z(x,y) = \sin\frac{n\pi x}{a} \; \sin\frac{m\pi y}{b} \quad , \quad n,m = 1,2,\ldots$$

$$-\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - k^2 + \left(\frac{\omega}{c}\right)^2 = 0 \Rightarrow k^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{a}\right)^2$$

↑ relativistic, massive, dispersion relation

~~for a given frequency~~

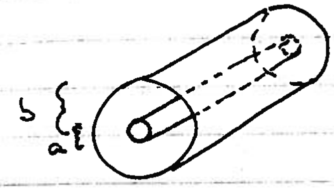
$$\omega = \sqrt{k^2c^2 + \underbrace{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2}_{\equiv \mu^2}}$$

for $\omega < \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2} \Rightarrow k$ is imaginary, wave doesn't propagate

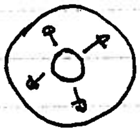
phase velocity: $v = \frac{\omega}{k} = \frac{\sqrt{k^2c^2 + \mu^2}}{k} > c$

group velocity: $v_g = \frac{d\omega}{dk} = \frac{1}{\cancel{2}\sqrt{k^2c^2+\mu^2}}\,\cancel{2}kc^2 < c$

EXAMPLE: TEM in a coaxial cable

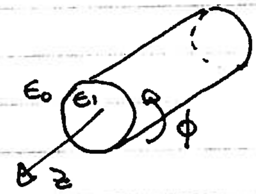


$$\mathcal{E}_z = \mathcal{B}_z = 0 \quad \Rightarrow \quad \underbrace{\begin{matrix}\nabla_\perp \times \mathcal{E}_\perp = 0 \\ \nabla_\perp \cdot \mathcal{E}_\perp = 0\end{matrix}}_{\text{electrostatic}} \qquad \underbrace{\begin{matrix}\nabla_\perp \times \mathcal{B}_\perp = 0 \\ \nabla_\perp \cdot \mathcal{B}_\perp = 0\end{matrix}}_{\text{magnetostatic}}$$



$$\mathcal{E}_\perp = \frac{A}{\rho}\,\hat{\rho} \qquad \mathcal{B}_\perp = \frac{A}{\rho}\,\hat{\phi}$$

EXAMPLE: dielectric waveguide (like an optical fiber)



Same as before but w/ different b.c.'s:

$$E \sim B \underset{r\to\infty}{\sim} 1/r^2 \text{ (TEM)} \quad \text{or} \quad e^{-\mu r} \text{ (TE, TM)}$$

TE modes w/ no ϕ dependence: $\mathcal{E}_z = 0$

$$\Big[\nabla_\perp^2 \underbrace{- k^2 + \epsilon_1\left(\frac{\omega}{c}\right)^2}_{\equiv +k_1^2}\Big]\mathcal{B}_z = 0 \,, \quad \rho < a$$

$$\Big[\nabla_\perp^2 \underbrace{- k^2 + \epsilon_0\left(\frac{\omega}{c}\right)^2}_{\equiv -\kappa_0^2}\Big]\mathcal{B}_z = 0 \,, \quad \rho > 0$$

$$\Downarrow$$

regular @ r=0

$$\mathcal{B}_z(\rho) = \begin{cases} J_0(k_1\rho) \,, & \rho < a \\ A\, K_0(\kappa_0\rho) \,, & \rho > 0 \end{cases}$$

only relative normalization matters

decays exp. @ r=∞

$$\nabla_\perp \times \mathcal{E}_\perp = \frac{i\omega}{c} B_z \Rightarrow \frac{1}{\rho}\frac{d}{d\rho}(\rho\, \mathcal{E}_\varphi) = \frac{i\omega}{c} B_z \,, \qquad \text{(i)}$$

$$ik\, \mathcal{E}_\perp = \frac{i\omega}{c}\hat{z}\times B_\perp \Rightarrow \underset{\text{(ii)}}{ik\, \mathcal{E}_\rho = -\frac{i\omega}{c} B_\varphi} \,, \quad ik\, \mathcal{E}_\varphi = \frac{i\omega}{c} B_\rho \quad \text{(iii)}$$

$$\nabla_\perp \times B_\perp = 0 \Rightarrow \frac{1}{\rho}\frac{d}{d\rho}(\rho B_\rho) = 0 \quad \text{(iv)}$$

n_1 or n_0 $\rightarrow$ n^2

$$ik B_\perp - \nabla_\perp B_z + \frac{i\omega}{c}\hat{z}\times\mathcal{E}_\perp = 0 \Rightarrow ik B_\varphi - \frac{1}{\rho}\cancelto{0}{\frac{\partial B_z}{\partial \varphi}} + \frac{i\omega n^2}{c}\mathcal{E}_\rho = 0 \quad \text{(v)}$$

$$ik B_\rho - \frac{\partial B_z}{\partial \rho} - \frac{i\omega n^2}{c}\mathcal{E}_\varphi = 0 \quad \text{(vi)}$$

curl in cylindrical coordinates
and no ϕ dependence

$$\nabla \times v = - \frac{\partial v_z}{\partial z}\hat{\rho} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_z}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho}\frac{d}{d\rho}(\rho v_z)\,\hat{z}$$

(iv) $\Rightarrow B_\rho \sim \frac{1}{\rho} \xrightarrow[\rho\to 0]{} \infty \Rightarrow B_\rho = 0$

(ii) $\Rightarrow \mathcal{E}_\rho = 0$

(iii) $\Rightarrow \mathcal{E}_\varphi = \frac{\omega}{kc} B_\rho$

$$\text{(vi)} \Rightarrow B_\rho = \frac{1}{k}\frac{\partial B_z}{\partial \rho} + \frac{\omega n^2}{kc}\underbrace{\frac{\omega}{kc}B_\rho}_{\mathcal{E}_\varphi} = \frac{1}{k}\begin{cases} -k_1 J_1(k_1\rho) \\ -A k_0 K_1(k_0\rho) \end{cases} + \frac{\omega^2 n^2}{k^2c^2} B_\rho$$

$$\text{or} \quad B_\rho \underbrace{\left(1 - \frac{\omega^2 n^2}{k^2 c^2}\right)}_{\frac{1}{k^2}\underbrace{\left(k^2 - \frac{\omega^2 n^2}{c^2}\right)}_{k_{1,0}^2}} = \frac{1}{k}\begin{cases} -k_1 J_1(k_1\rho) \\ -A k_0 K_1(k_0\rho) \end{cases}$$

$$B_\rho = \frac{k}{k_{1,0}^2}\frac{d}{d\rho} B_z$$

$$B_z(\rho=a+) = B_z(\rho=a^-) \Rightarrow \quad J_0(\kappa_1 a) = A\, K_0(\kappa_0 a)$$

$$\underbrace{E_\varphi(\rho=a^+)}_{\frac{\omega}{kc} B_\rho} = \underbrace{E_\varphi(\rho=a^-)}_{\frac{\omega}{kc} B_\rho} \Rightarrow \quad \frac{1}{\kappa_1^2}\cancel{\kappa_1}\, J_1(\kappa_1 a) = \frac{A}{\kappa_0^2}\cancel{\kappa_0}\, K_1(\kappa_0 a)$$

A and κ determined from eqs. above.

$$\kappa_1 \frac{J_0(\kappa_1 a)}{J_1(\kappa_1 a)} = \kappa_0 \frac{K_0(\kappa_0 a)}{K_1(\kappa_0 a)}$$

$$\kappa_0^2 = k^2 - \frac{n_0^2\omega^2}{c^2}$$

$$\kappa_1^2 = -k^2 + n_1^2\frac{\omega^2}{c^2}$$

$$\kappa_0^2 + \kappa_1^2 = (n_1^2 - n_0^2)\frac{\omega^2}{c^2} = (\epsilon_1 - 1)\frac{\omega^2}{c^2}$$