RADIATION

RADIATION BY A POINT CHARGE

$$\partial_\mu \underbrace{F^{\mu\nu}}_{\partial_\mu A_\nu - \partial_\nu A_\mu} = \frac{4\pi}{c} J^\nu$$

In the Lorentz gauge

$$\underbrace{\partial_\mu \partial^\mu}_{\Box} A_\nu = \frac{4\pi}{c} J_\nu$$

$$\Downarrow$$

$$\boxed{\begin{array}{l} A_\mu(x) = \frac{1}{c}\int d^4y \;\; G(x-y)\, J_\mu(y) \;\; + A^0_\mu(x) \\ \text{with} \\ \qquad \partial_\mu \partial^\mu G(x-y) = 4\pi\, \delta(x-y) \\ \qquad \partial_\mu \partial^\mu A^0_\mu(x) = 0 \end{array}}$$

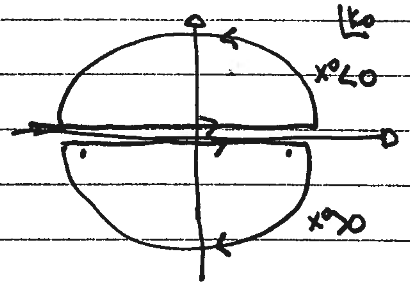
$$\left\{\begin{array}{l} A_\mu(x) \\ J_\mu(x) \\ G(x) \end{array}\right. = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} \left\{\begin{array}{l} A_\mu(k) \\ J_\mu(k) \\ G(k) \end{array}\right.$$

$$-k_\mu k^\mu \, G(k) = 4\pi \quad \Rightarrow \quad G(k) = -\frac{4\pi}{k^2}$$

G(x) in position space requires a little more work:

$$G(x) = 4\pi\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik.x}}{k^2} = -4\pi\int\frac{d^3k}{(2\pi)^3}\int\frac{dk^0}{2\pi}\,\frac{e^{-ik_0x^0+i\vec{k}\cdot\vec{x}}}{k_0^2-\vec{k}^2}$$

This integral is ill defined (simple poles at $k_0 = \pm|\vec{k}|$). We can make it well defined by adding a small imaginary part to $|\vec{k}|$ or, what's the same, deform the contour to avoid the poles. For instance:



$$G_R(x) = -4\pi\int\frac{d^3k}{(2\pi)^3}\int dk_0\,\frac{e^{-ik_0x^0+i\vec{k}\cdot\vec{x}}}{\underbrace{(k_0+i0)^2-|\vec{k}|^2}_{[k_0+i0-|k|][k_0+i0+|k|]}}$$

$$= -4\pi\,\Theta(x^0)\int\frac{d^3k}{(2\pi)^3}\,\frac{e^{i\vec{k}\cdot\vec{x}}}{2|\vec{k}|}\,\frac{(-2\pi i)}{2\pi}\underbrace{\left[e^{-i|k|x^0}-e^{i|k|x^0}\right]}_{-2i\sin|k|x^0}$$

$$= +4\pi\,\Theta(x^0)\int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{x}}\,\frac{\sin|k|x^0}{|k|}$$

$$\cancel{= -4\pi\,\Theta(x^0)\int\frac{dk_z}{(2\pi)^3}\,e^{ik_zz}\,\frac{\sin\sqrt{k_z^2+k_\perp^2}\,x^0}{\sqrt{k_z^2+k_\perp^2}}}$$

$$= 4\pi\,\Theta(x^0)\int_0^\infty\frac{dk\,k^2}{(2\pi)^3}\underbrace{\int_{-\pi}^{\pi}d\theta\,2\pi\sin\theta\;e^{ikx\cos\theta}}_{2\pi\int_{-1}^{1}d\cos\theta\;e^{ikx\cos\theta}}\;\frac{\sin kx^0}{k}$$

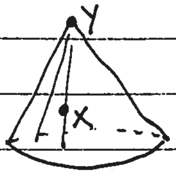
$$= \frac{8\pi^2}{8\pi^3}\,\Theta(x^0)\int_0^\infty dk\,k^2\,\frac{\cancel{2}\sin kx}{\cancel{i}kx}\,\frac{\sin kx^0}{k}$$

$$= \frac{1}{\pi}\,\Theta(x^0)\,\frac{1}{(2i)|\vec{x}|i}\int_0^\infty dk\,\left(e^{ikx}-e^{-ikx}\right)\left(e^{ikx^0}-e^{-ikx^0}\right)$$

$$= \frac{\Theta(x^0)}{|\vec{x}|}\left(-\frac{1}{2\pi}\right)\underbrace{\int_0^\infty dk\left[e^{ik(x+x^0)}+e^{-ik(x+x^0)}-e^{-ik(x-x^0)}-e^{ik(x-x^0)}\right]}_{\int_{-\infty}^{\infty}dk\left[e^{ik(x+x^0)}-e^{-ik(x-x^0)}\right]}$$

$$= -\frac{1}{2\pi}\,\frac{\Theta(x^0)}{|\vec{x}|}\,2\pi\left(-\delta(x-x^0)+\underbrace{\delta(x+x^0)}_{=0\ (|x|,\,x^0>0)}\right) = \frac{\Theta(x^0)}{|\vec{x}|}\,\delta(x-x^0)$$

$$G_R(x-y) = \frac{\overset{x^0 > y^0 \text{ (retarded)}}{\theta(x^0-y^0)}}{|\vec{x}-\vec{y}|}\,\underbrace{\delta(x^0-y^0-|\vec{x}-\vec{y}|)}_{\text{only the charge/curents on the past light-cone influence the field}} = 2\,\theta(x^0-y^0)\,\delta\big(\underbrace{(x^0-y^0)^2-|\vec{x}-\vec{y}}_{(x-y)^2}$$



A different "i0" prescription amounts to a different $A^\mu_0(x)$ to be added in order to satisfy boundary/initial/final conditions.

$$G_A(k) = -\frac{4\pi}{(k_0-i0)^2-|k|^2} = -\frac{4\pi}{k_0^2-|k|^2-i0k_0} = -4\pi\left[P\frac{1}{k_0^2-\vec{k}^2} + i\pi\,\mathrm{sgn}(k_0)\,\delta(k_0^2-\vec{k}^2)\right]$$

$$G_R(k) - G_A(k) = -4\pi\,(-2\pi i)\,\mathrm{sgn}(k_0)\,\delta(k_0^2-\vec{k}^2)$$

$$\Downarrow$$

$$G_R(x) - G_A(x) = 4\pi\,(2\pi i)\int\frac{dk_0\,d^3k}{(2\pi)^4}\,\mathrm{sgn}(k_0)\,\delta(k_0^2-\vec{k}^2)\,e^{-i(k_0x^0-i\vec{k}\cdot\vec{x})}$$

$$= 4\pi i\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2|k|}\,e^{-i(kx^0-\vec{k}\cdot\vec{x})} = \text{plane wave superposition}$$

toy model: forced h.o.

$$\ddot{x}(t) + \omega_0^2 x(t) = \frac{f(t)}{m} \Rightarrow x(t) = \int dt'\,G(t-t')\frac{f(t')}{m} + x_0(t)$$

$$\partial_t^2 G(t) + \omega_0^2 G(t) = \delta(t-t')$$

$$(-\omega^2+\omega_0^2)\,G(\omega) = \Rightarrow G(\omega) = \frac{1}{\omega_0^2-\omega^2}$$

$$G_R(t) = -\int\frac{d\omega}{2\pi}\,\frac{e^{-i\omega t}}{(\omega+i0)^2-\omega_0^2} = +i\,\theta(t)\left[\frac{e^{-i\omega_0 t}}{2\omega} - \frac{e^{i\omega_0 t}}{2\omega}\right]$$

$$= -\theta(t)\sin\omega_0 t$$

$$x(t) = -\int_{-\infty}^{\infty}dt'\,\theta(t-t')\sin\omega_0(t-t')\frac{f(t')}{m} + x_0(t)$$

$$x(-\infty) = -\underbrace{\int_{-\infty}^{-\infty}dt'\,\sin\omega_0(t_0-t')\frac{f(t')}{m}}_{=0} + x_0(-\infty) \Rightarrow$$

G_R convenient when initial conditions are prescribed

LIÉNARD-WIECHERT POTENTIAL (field due to a point charge)

$$J^\mu(x) = q c \int d\tau \; \underset{\text{proper time}}{\frac{dx^\mu}{d\tau}} \; \underset{\text{particle trajectory}}{\delta(x - r(\tau))}$$

$$= q c \int \underbrace{\frac{dt}{\gamma}}_{d\tau} \; \gamma \underbrace{\frac{dr^\mu}{d\tau}}_{(c,\, \dot{\vec{r}})} \; \underbrace{\delta(x - r(\tau))}_{\underbrace{\delta(x^0 - r^0(\tau))}_{\substack{\delta(x^0 - ct) \\ = \frac{1}{c}\delta\left(\frac{x^0}{c} - t\right)}} \; \delta(\vec{x} - \vec{r}(t))}$$

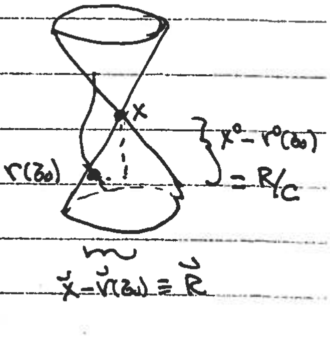
$$= q \left(c, \vec{r}(x^0/c)\right) \delta\left(\vec{x} - \vec{r}(x^0/c)\right)$$

$$A_\mu(x) = \frac{1}{c} \int d^4y \; G_R(x-y) \; J_\mu(y)$$

$$= q\, 2 \int d\tau \int d^4y \; \Theta(x^0 - y^0) \; \delta\left((x-y)^2\right) \frac{dr_\mu}{d\tau} \; \delta(y - r(\tau))$$

$$= 2q \int d\tau \; \Theta(x^0 - r^0(\tau)) \; \underbrace{\delta\left((x - r(\tau))^2\right)}_{\frac{1}{\left|-2(x - r(\tau)) \cdot \frac{dr}{d\tau}\right|} \delta(\tau - \tau_0)} \; \frac{dr_\mu}{d\tau}$$

$$= q \; \frac{dr_\mu}{d\tau} \; \frac{1}{(x - r(\tau)) \cdot \frac{dr}{d\tau}} \Bigg|_{\tau = \tau_0}$$



non-relativistic notation:

$$r^\mu = (ct, \vec{r}(t))$$

$$\frac{dr^\mu}{d\tau} = \gamma \frac{dr^\mu}{dt} = (\gamma c, \gamma \vec{v})$$

$$x - r(\tau_0) = (ct - c\tau_0, \vec{x} - \vec{r}(t_0)) \equiv (R, \vec{R})$$

$$(x - r(\tau)) \cdot \frac{dr}{d\tau} = R\gamma c - \gamma \vec{R}\cdot\vec{v} = Rc\gamma\left(1 - \frac{\vec{v}}{c}\cdot\hat{R}\right)$$

$$A_0 = \phi = \left.\frac{q\,\cancel{\gamma}\cancel{c}}{R\cancel{c}\cancel{\gamma}\left(1 - \frac{\vec{v}}{c}\cdot\hat{R}\right)}\right|_{\text{ret}} \xrightarrow[v \to 0]{} \frac{q}{R}$$

$$\vec{A} = \left.\frac{q\,\cancel{\gamma}\,\vec{v}}{Rc\cancel{\gamma}\left(1 - \frac{\vec{v}}{c}\cdot\hat{R}\right)}\right|_{\text{ret.}} \xrightarrow[v \to 0]{} \frac{q\vec{v}}{Rc}$$

field strength: ∂_μ hits on x and implicitly on τ_0; it's better to take a step back.

$$\partial_\mu A_\nu(x) = 2q \int d\tau \int d^4y \; \underbrace{\partial_\mu\, \theta(x^0 - y^0)\, \delta((x-y)^2)}_{\underset{0}{\delta(x^0 - y^0)\,\delta(-R^2)} + \theta(x^0-y^0)\,\partial_\mu \delta((x-y)^2)} \frac{dr_\nu}{d\tau} \; \delta(y - r(\tau))$$

$$= 2q \int d\tau \; \theta(x^0 - r(\tau)) \; \partial_\mu\, \delta((x-r)^2) \frac{dr_\nu}{d\tau}$$

$$\text{but } \partial_\mu\, \delta((x - r(\tau))^2) = \partial_\mu (x - r(\tau))^2 \underbrace{\frac{d\tau}{d(x-r)^2}}_{\left[\frac{d(x-r)^2}{d\tau}\right]^{-1}} \frac{d}{d\tau}\, \delta((x - r(\tau))^2)$$

$$= 2(x - r(\tau))_\mu \left[-2(x - r(\tau))\cdot\frac{dr}{d\tau}\right]^{-1} \frac{d}{d\tau}\, \delta((x - r(\tau))^2)$$

⑥

$$\partial_\mu A_\nu(x) = -2q \int d\tau\, \theta(x^a - r(\tau))\, \frac{(x-r(\tau))_\mu}{(x-r(\tau))\cdot\frac{dr}{d\tau}}\, \frac{dr_\nu}{d\tau}\, \frac{d}{d\tau}\, \delta((x-r(\tau))^2)$$

$$= 2q \int d\tau\, \underset{\text{no } d/d\tau \text{ here because} \sim \delta(x^0 - r^0(\tau))\,\delta(-\vec{R}) = 0}{\theta(x^0 - r(\tau))}\, \frac{d}{d\tau}\left[(x-r(\tau))_\mu \frac{dr_\nu}{d\tau} \frac{1}{(x-r(\tau))\cdot\frac{dr}{d\tau}} \right] \delta((x-r(\tau))^2)$$

$$= q\, \frac{1}{(x-r(\tau))\cdot v}\, \frac{d}{d\tau}\left[\frac{(x-r(\tau))_\mu v_\nu}{(x-r(\tau))\cdot v} \right]\Bigg|_{\text{ret.}}$$

and

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{q}{(x-r(\tau))\cdot v}\, \frac{d}{d\tau}\left[\frac{(x-r(\tau))_\mu v_\nu - (x-r(\tau))_\nu v_\mu}{(x-r(\tau))\cdot v} \right]_{\text{ret.}}}$$

now:

$$(x-r)^\mu = (ct - ct_0, \vec{x} - \vec{r}(t_0)) = (R, \vec{R})$$

$$\frac{d\vec{R}}{d\tau} = \gamma \frac{d\vec{R}}{dt} = \gamma \vec{v}, \qquad \frac{d}{dt}\gamma = \frac{d}{dt}\frac{1}{\sqrt{1-v^2/c^2}} = \frac{\gamma^3 v\cdot\dot{v}}{c^2}$$

$$v^\mu = (\gamma c, \gamma\vec{v}), \qquad \frac{dv^\mu}{d\tau} = \gamma\frac{d}{dt}(\gamma c, \gamma\vec{v}) = \left(\frac{\gamma^4}{c} v\cdot\dot{v},\ \gamma^2 \dot{\vec{v}} + \gamma^4 \frac{\vec{v}\,(v\cdot\dot{v}}{c^2}\right.$$

$$\frac{d}{d\tau}\left[(x-r(\tau))\cdot v\right] = \underbrace{\frac{d}{d\tau}(x-r(\tau))\cdot v}_{-v\cdot v} + (x-r(\tau))\cdot\frac{dv}{d\tau}$$

$$= -c^2 + (x-r(\tau))\cdot\frac{dv}{d\tau}$$

$$(x-r(\tau))\cdot v = \gamma c R - \gamma\,\vec{v}\cdot\vec{R} = \gamma c R\left(1 - \frac{\vec{v}}{c}\cdot\hat{R}\right) = \gamma c R\,(1 - \vec{\beta}\cdot\hat{R})$$

$$F_{\mu\nu} = \frac{q}{(x-r)\cdot v}\left[\frac{-v_\mu \cancel{v_\nu} + (x-r)_\mu\, dv^\nu/d\tau + v_\nu \cancel{v_\mu} - (x-r)_\nu\, dv^\mu/d\tau}{(x-r)\cdot v}\right.$$

$$\left. - \frac{1}{[(x-r)\cdot v]^2}\left(-c^2 + (x-r)\cdot\frac{dv}{d\tau}\right)\left((x-r)_\mu v_\nu - (x-r)_\nu v_\mu\right)\right]$$

$$\vec E = F_{0i}\,\hat e_i = -\frac{q}{[\gamma c R(1-\vec\beta\cdot\hat R)]^2}\left[R\left(\gamma^2\dot{\vec v} + \gamma^4\frac{v\cdot\dot v}{c^2}\vec v\right) - \vec R\left(\frac{\gamma^4}{c}v\cdot\dot v\right)\right]$$

$$- \frac{q}{[\gamma c R(1-\vec\beta\cdot\hat R)]^3}\left[\left(c^2 - R\frac{\gamma^4}{c}v\cdot\dot v - \vec R\cdot\left(\gamma^2\dot{\vec v} + \frac{\gamma^4}{c^2}v\cdot\dot v\,\vec v\right)\right)\left(R\gamma\vec v - \vec R\gamma c\right)\right]$$

$$= -\frac{q}{[\gamma c R(1-\vec\beta\cdot\hat R)]^3}\,c^3 R\gamma\left(\frac{\vec v}{c} - \hat R\right) - \frac{q}{[\gamma c R(1-\vec\beta\cdot\hat R)]^2}\left[R\gamma^2\dot{\vec v} + \frac{R\gamma^4}{c}v\cdot\dot v\,(\vec\beta - \hat R\right.$$

$$- \frac{q}{[\gamma c R(1-\vec\beta\cdot\hat R)]^3}\left[R\gamma c(\vec\beta - \hat R)\left(\frac{R\gamma^4}{c}v\cdot\dot v\,(-1 + \vec\beta\cdot\hat R) - \gamma^2\vec R\cdot\dot{\vec v}\right)\right]$$

$$= \frac{q}{\gamma^2 R^2(1-\vec\beta\cdot\hat R)^3}(\hat R - \vec\beta)$$

$$+ \frac{q}{\cancel{\gamma^3}R^3(1-\vec\beta\cdot\hat R)^3}\left[(+1 - \vec\beta\cdot\hat R)\,\cancel{R}\,\overset{2}{\cancel{\gamma^5}}\,\cancel{c}\,\beta\cdot\dot\beta\,(\vec\beta - \hat R) + \cancel{\gamma^3}\cancel{R}\cancel{c}\,(\vec\beta-\hat R)\,\hat R\cdot\dot{\vec\beta}\right.$$

$$\left. - \left(\cancel{\gamma^3}\cancel{c}\cancel{R}\,\dot{\vec\beta} + \overset{2}{\cancel{\gamma^5}}\cancel{R}\cancel{c}\,\beta\cdot\dot\beta\,(\vec\beta-\hat R)\right)(1-\vec\beta\cdot\hat R)\right]$$

$$= \frac{q}{\gamma^2R^2(1-\vec\beta\cdot\hat R)}(\hat R - \vec\beta) + \frac{q}{\gamma R(1-\vec\beta\cdot\hat R)^3}\left[\cancel{(1-\vec\beta\cdot\hat R)\gamma^2(\vec\beta-\hat R)\beta\cdot\dot\beta} + (\vec\beta-\hat R)\,\hat R\cdot\dot{\vec\beta}\right.$$

$$\left. - \left(\dot{\vec\beta} + \cancel{\gamma^2\beta\cdot\dot\beta\,(\vec\beta-\hat R)}\right)(1-\vec\beta\cdot\hat R)\right]$$

$$= \frac{q}{\gamma^2R^2(1-\vec\beta\cdot\hat R)}(\hat R - \vec\beta) + \frac{q}{c\gamma R(1-\vec\beta\cdot\hat R)^3}\underbrace{\left[\hat R\cdot\dot{\vec\beta}\,(\vec\beta-\hat R) - (1-\vec\beta\cdot\hat R)\,\dot{\vec\beta}\right]}_{\hat R\times\left((\hat R-\vec\beta)\times\dot{\vec\beta}\right)}$$

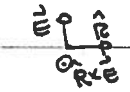
$$\vec{E} = \underbrace{\frac{q}{R^2}}_{\text{near field}} \frac{(\hat{R}-\hat{\beta})}{\gamma^2(1-\vec{\beta}\cdot\hat{R})^3}\Big|_{ret} + \underbrace{\frac{q}{cR}}_{\sim 1/R\ :\ \text{radiation field}} \frac{1}{(1-\vec{\beta}\cdot\hat{R})^3}\, \hat{R}\times(\hat{R}-\hat{\beta})\times\underbrace{\dot{\vec{\beta}}}_{\text{acceleration}}\Big|_{ret}$$

similarly $\vec{B} = \hat{R}\times\vec{E}$

Total power radiated:

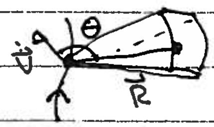
non-relativistic case: $\underbrace{\vec{E}}_{\text{only the radiation part}} = \frac{q}{c}\frac{\hat{R}\times(\hat{R}\times\dot{\vec{\beta}})}{R}\Big|_{ret}$ ← polarized in the $\hat{R}, \dot{\vec{\beta}}$ plane

$\vec{E}\cdot\hat{R} = 0$

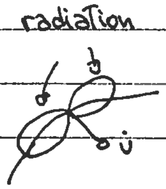


$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\vec{E}\times(\hat{R}\times\vec{E}) = \frac{c}{4\pi}|E|^2\,\hat{R}$$

$$\underset{\text{solid angle}}{\overset{\text{power}}{\frac{dP}{d\Omega}}} = \frac{c}{4\pi}\frac{q^2}{c^2}\frac{1}{R^2}\underbrace{R^2}_{d\Omega R^2 \text{ is the area}}\left|\hat{R}\times(\hat{R}\times\frac{\dot{\vec{v}}}{c})\right|^2$$



$$= \frac{q^2}{4\pi c^3}|\dot{v}|^2\sin^2\theta$$



$$P = \int d\Omega\,\frac{dP}{d\Omega} = 2\cancel{\pi}\underbrace{\int_0^\pi d\theta\,\sin^3\theta}_{4/3}\frac{q^2}{4\cancel{\pi}c^3}|\dot{v}|^2$$

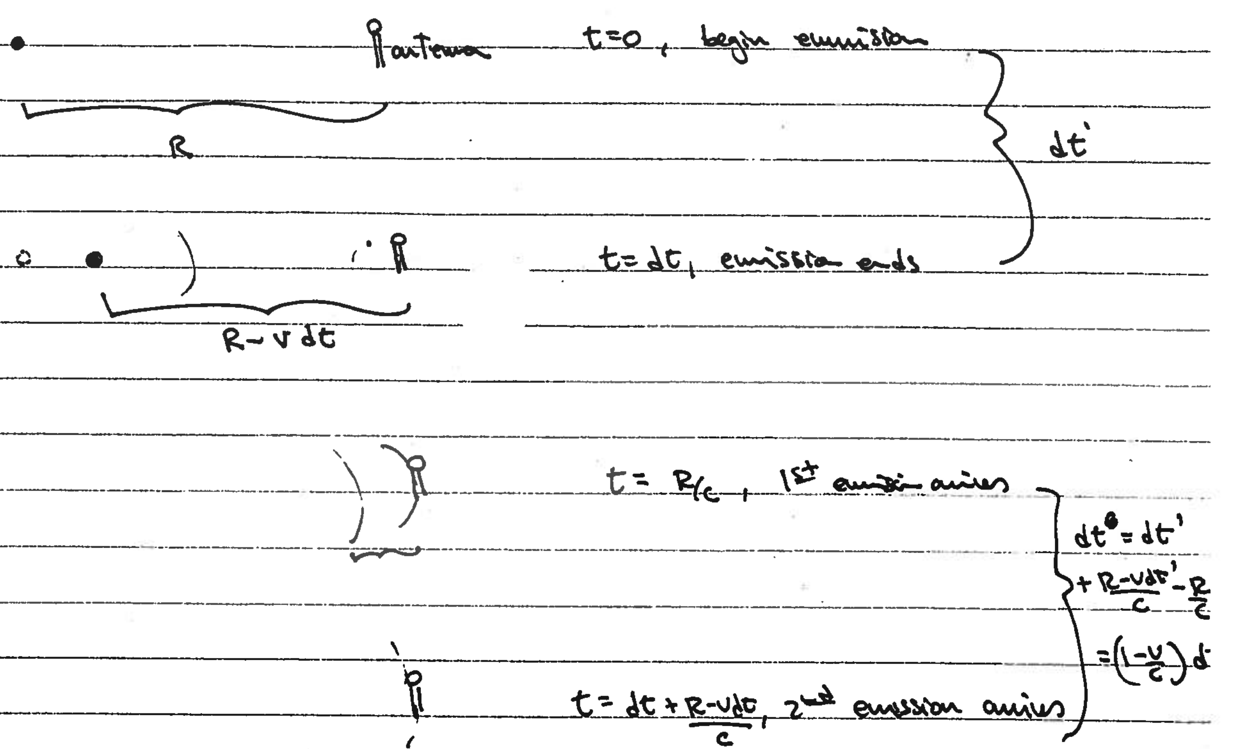
$$= \frac{2}{3}\frac{q^2}{c^3}|\dot{v}|^2$$

(Larmor's formula)

relativistic case:

There's a difference between the power measured by a static observer ($\sim S = \frac{c}{4\pi} E \times B$) and the power emitted by the particle. It's <u>not</u> a relativistic effect, but it's $\sim v/c$. It's analogous to the Doppler effect.

<u>All Times measured by the static observer</u>



$dt^{\theta} = dt' + \frac{R-vdt'}{c} - \frac{R}{c} = \left(1-\frac{v}{c}\right)dt'$

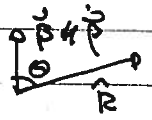
power emitted by the particle in unit of time

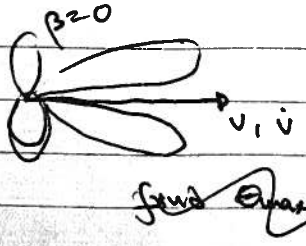
$$\frac{dP(t')}{d\Omega} = \frac{c}{4\pi} R^2 \vec{S}\cdot\hat{R}\,\frac{dt}{dt'} = \frac{c}{4\pi} R^2 \frac{(1-\vec{\beta}\hat{R})}{\cancel{}}\vec{E}\times(\hat{R}\times\vec{E})\cdot\hat{R}$$

$$= \frac{c}{4\pi}R^2 \overset{(1-\vec{\beta}\hat{R})}{\cancel{}}\frac{|E|^2}{\cancel{}} = \frac{q^2}{4\pi c}\frac{\left(\hat{R}\times((\hat{R}-\vec{\beta})\times\dot{\vec{\beta}})\right)^2}{(1-\vec{\beta}\cdot\hat{R})^5}$$

<u>$\vec{\beta}$ and $\dot{\vec{\beta}}$ parallel</u>

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c}\frac{\dot{v}^2}{c^2}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$





~~for $\beta \to 1$, $\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c^3}\dot{v}^2\frac{\sin^2\theta}{(1-\cos\theta)^5}$~~

~~find $\theta_{max} \Rightarrow \frac{2\sin\theta\cos\theta}{(1-\cos\theta)^5} - \frac{5\sin^3\theta}{(1-\cos\theta)^6} = 0$~~

$$[illegible] = \frac{q^2\dot{v}^2}{4\pi c^3}2\pi\underbrace{\int_0^\pi d\theta\frac{\sin^3\theta}{(1-\beta\cos\theta)^5}}_{\frac{4}{3}\gamma^6}$$

$$= \frac{2q^2}{3c^3}\dot{v}^2\gamma^6$$

maxim of P :

$$\frac{d}{d\theta}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\Bigg|_{\theta=\theta_{max}}=0 \Rightarrow \cancel{\frac{2\sin\theta\cos\theta}{(\quad)^5}}+\frac{5\sin^2\theta(-\beta\cancel{\sin\theta})}{(\quad)^{\cancel{6}}}=0$$

$$2\cos\theta = \frac{5\beta\sin^2\theta}{1-\beta\cos\theta}=\frac{5\beta(1-\cos^2\theta)}{1-\beta\cos\theta}$$

~~$5\beta\cos^2\theta+2\cos\theta-5\beta=0$~~

~~$\cos^2\theta+\frac{2}{5\beta}\cos\theta-1=0$~~

~~$\cos\theta_{max}=-\frac{2}{2\cdot5\beta}\pm\frac{1}{2}\sqrt{\frac{4}{25\beta^2}+4}$~~

~~$=-\frac{1}{5\beta}\pm\sqrt{\frac{1+25\beta^2}{25\beta^2}}$~~

~~$=-\frac{1}{5\beta}$~~

$$2\cos\theta-2\beta\cos^2\theta = 5\beta+5\beta\cos^2\theta=0$$

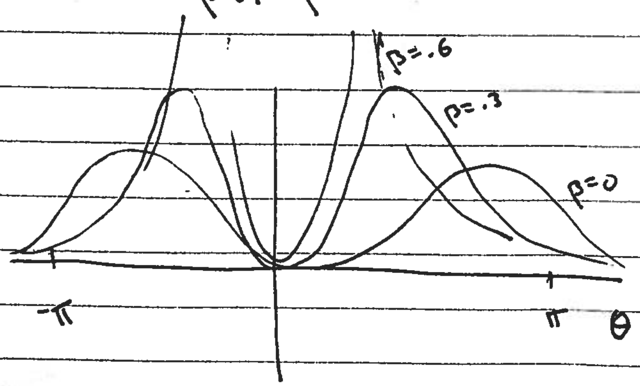
$$3\beta\cos^2\theta+2\cos\theta-5\beta=0$$

$$\cos^2\theta+\frac{2}{3\beta}\cos\theta-\frac{5}{3}=0$$

$$\cos\theta_{max}=\frac{-2}{2\cdot3\beta}\pm\frac{1}{2}\sqrt{\frac{4}{9\beta^2}+\frac{4\cdot5}{3}}=-\frac{1}{3\beta}\pm\sqrt{\frac{1}{9\beta^2}+\frac{5}{3}}$$

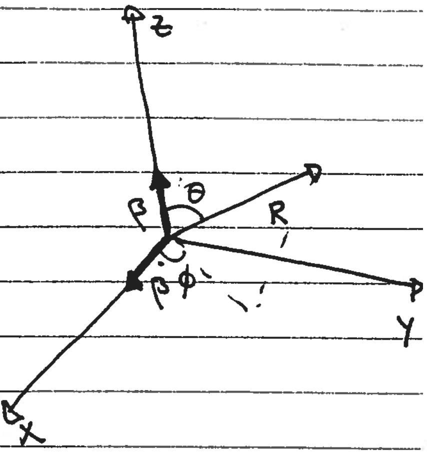
$$=-\frac{1}{3\beta}\pm\sqrt{\frac{1+15\beta^2}{9\beta^2}}=\frac{1}{3\beta}\left[-1\pm\sqrt{1+15\beta^2}\right]$$

$$\underset{\beta\to1}{\simeq}\frac{1}{\beta}, \quad \theta_{max}=\arccos 1/\beta \xrightarrow[\beta\to1]{} 0$$



circular motion $\vec{\beta} \perp \dot{\vec{\beta}}$ (cyclotron (NR) and synchrotron (R) radiation)

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{R} \times (\hat{R}-\vec{\beta}) \times \dot{\vec{\beta}} \right|^2}{(1-\vec{\beta}\cdot\hat{R})^5}$$



$$\vec{\beta} = \beta \hat{z}$$

$$\dot{\vec{\beta}} = \dot{\beta} \hat{x}$$

$$\hat{R} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$1-\vec{\beta}\cdot\hat{R} = 1-\beta\cos\theta$$

$$\hat{R}-\vec{\beta} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + (\cos\theta-\beta)\hat{z}$$

$$(\hat{R}-\vec{\beta})\times\dot{\vec{\beta}} = \dot{\beta}\left[-\sin\theta\sin\phi\,\hat{z} + (\cos\theta-\beta)\hat{y}\right]$$

$$\hat{R}\times(\hat{R}-\vec{\beta})\times\dot{\vec{\beta}} = \dot{\beta}\begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta-\beta & -\sin\theta\sin\phi \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix}$$

$$= \dot{\beta}\Big[\left(-\sin^2\theta\sin^2\phi - \cos\theta(\cos\theta-\beta)\right)\hat{x} - \sin^2\theta\sin\phi\cos\phi\,\hat{y}$$

~~[illegible]~~ $$+ \sin\theta\cos\phi(\cos\theta-\beta)\,\hat{z}\Big]$$

$$\frac{1}{\dot{\beta}}\left|\hat{R}\times(\hat{R}-\vec{\beta})\times\dot{\vec{\beta}}\right|^2 = \sin^4\theta\sin^4\phi + \cos^2\theta(\cos\theta-\beta)^2 + 2\sin^2\theta\sin^2\phi\cos\theta(\cos\theta-\beta)$$
$$+ \sin^4\theta\sin^2\phi\cos^2\phi + \sin^2\theta\cos^2\phi(\cos\theta-\beta)^2$$

$$= \sin^4\theta\sin^2\phi\ \cancel{+ (\cos\theta-\beta)^2(\cos^2\theta + \sin^2\theta\cos^2\phi) + 2\cos\theta(\cos\theta-\beta)\sin^2\theta\sin^2\phi}$$
$$+ \cos^2\theta(\cos\theta-\beta)^2 + 2\sin^2\theta\cos\theta(\cos\theta-\beta)$$
$$+ \cos^2\phi\sin^2\theta\left((\cos\theta-\beta)^2 - 2\cos\theta(\cos\theta-\beta)\right)$$

⑫

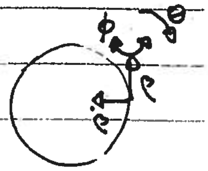
$$\underbrace{\left(\sin^2\theta + \cos\theta(\cos\theta-\beta)\right)^2 = (1-\beta\cos\theta)^2}$$

$$= \sin^4\theta + \cos^2\theta\,(\cos\theta-\beta)^2 + 2\sin^2\theta\,\cos\theta(\cos\theta-\beta)$$

$$+ \cos^2\phi\,\sin^2\theta\,\underbrace{\underbrace{\left(-\sin^2\theta + (\cos\theta-\beta)^2 - 2\cos\theta(\cos\theta-\beta)\right)}_{-1+\cancel{\cos^2\theta}-\cancel{\cos^2\theta}+\beta^2-2\cancel{\beta\cos\theta}+2\cancel{\beta\cos\theta}}}_{-\frac{1}{\gamma^2}}$$

$$= (1-\beta\cos\theta)^2 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2}$$

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c^3}\frac{\dot{v}^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$



$$P(t') = \frac{q^2}{4\pi c^3}\,\dot{v}^2\,\underbrace{\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]\frac{1}{(1-\beta\cos\theta)^3}}_{\frac{8\pi\gamma^4}{3}}$$

$$= \frac{2q^2}{3c^3}\dot{v}^2\gamma^4$$ (Two powers of γ smaller than $\beta \parallel \dot{\beta}$)

but

$$= \frac{2q^2}{3c^3}\left(\frac{dp}{dt}\right)^2\frac{\gamma^4}{m^2} = \frac{2q^2}{3c^3 m^2}\gamma^2\left(\frac{dp}{dt}\right)^2$$ for $v \perp \dot{v}$

bigger by γ^2 for the same force

$p = \gamma m v$

$\frac{dp}{dt} = \gamma m\dot{v} + \dot{\gamma} m v$

$= \gamma m\dot{v} + \frac{v}{c^2}v\cdot\dot{v}\,m\gamma^3$

$= \begin{cases}\gamma^3 m\dot{v} & (v \parallel \dot{v}) \\ \text{or } \gamma m\dot{v} & (v \perp \dot{v})\end{cases}$

It's harder to accelerate in the v direction because the particle energy has to grow

compare To

$$P(t') = \frac{2q^2}{3c^3}\dot{v}^2\gamma^6 =$$

$$= \frac{2q^2}{3c^3}\frac{\cancel{\gamma^6}}{m^2\cancel{\gamma^6}}\left(\frac{dp}{dt}\right)^2$$ for $v_\parallel \dot{v}$

in synchrotrons:

for fixed energy $E = \gamma mc$, $\gamma \sim \frac{1}{Emc}$, $\gamma_{\text{electron}} \simeq 2000\, \gamma_{\text{proton}}$

for the same force ($\bar{e}$ and p have the same charge) $\frac{P_{\text{proton}}}{P_{\text{electron}}} = \frac{1}{\cancel{m^2}}\frac{\cancel{m^2}}{\gamma^2} \to$ $\bar{e}$ for synchrotrons is prohibitive

numerical examples:

linear ~~motion~~ accelerator: $P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2$

$$\frac{dp}{dt} = \frac{d}{dt}\gamma m v = \gamma m\frac{dv}{dt} - \frac{m}{2}\gamma^3(-2)\frac{v\cdot\dot{v}}{c^2}v = \gamma m a\underbrace{\left[1 + \frac{v^2\gamma^2}{c^2}\right]}$$

$$= \gamma ma\left[1 + \frac{v^2}{c^2}\frac{1}{1-v^2/c^2}\right] = \gamma ma\left[1 + \frac{1}{\frac{c^2}{v^2}-1}\right] \qquad \cancel{\frac{\gamma^2+v^2\gamma^2/c^2}{\gamma} = \frac{1}{\gamma}\left(\frac{1}{1-v^2/c^2}+\frac{v^2}{c^2}\right)}$$

$$= \gamma ma\,\frac{c^2/v^2 \cancel{-1} \cancel{+1}}{c^2/v^2-1} = ma\gamma^3 \qquad \cancel{= \frac{1}{\gamma}\left(\frac{1+v^2/c^2}{1-v^2/c^2}\right)}$$

$$\frac{dE}{dx} = \frac{d\,\gamma mc^2}{dt}\underbrace{\frac{dt}{dx}}_{1/v} = \frac{mc^2}{\cancel{v}}\left(\cancel{\frac{-1}{2}}\right)\gamma^3\left(\cancel{-2}\frac{\cancel{v}\cdot\dot{v}}{\cancel{c^2}}\right) = \gamma^3 ma$$

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2$$

power radiated →

$$\frac{P}{\underbrace{dE/dt}_{\frac{dE}{dx}\frac{dx}{dt}}} = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2\frac{1}{\frac{dE}{dx}v} = \underbrace{\frac{2}{3}\frac{e^2}{m^2c^3}\frac{1}{v}}\frac{dE}{dx} \sim 1 \Rightarrow$$

← power supplied by external forces

$$\underset{v\to c}{\simeq} \frac{2}{3}\frac{e^2/mc^2}{\underbrace{mc^2}_{\bar{e}\text{ rest mass}}} \qquad \frac{dE}{dx} \simeq \frac{\overset{\bar{e}\text{ rest mass}}{\overbrace{mc^2}}}{\underbrace{e^2/mc^2}_{3\times10^{-13}\text{ cm, classical }\bar{e}\text{ radius}}}$$

⇨ radiation losses are not a problem w/ linear accelerators

synchrotron: $v = 2\pi R\underbrace{\nu}_{\omega/2\pi} = R\omega \Rightarrow \beta = \frac{R\omega}{c}$, $a = v^2/R$

orbit radius

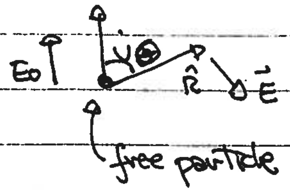
$$P = \frac{2}{3}\frac{e^2}{c^3}\gamma^4 a^2 = \frac{2}{3}\frac{e^2}{c^3}\underbrace{\frac{v^4}{R^2}}_{\frac{c^4}{R^2}\beta^4}\gamma^4 = \frac{2}{3}\frac{e^2c\,\beta^4}{R^2}\gamma^4$$

$$\frac{E}{\text{turn}} = \underbrace{\frac{2}{3}\frac{e^2 c}{m^2}\frac{\beta^4}{R^2}\gamma^4}_{\text{power radiated}} \underbrace{\frac{2\pi R}{c\beta}}_{\text{period}} = \frac{4\pi}{3}\frac{e^2}{m^2}\frac{\beta^3}{R}\gamma^4 \simeq 10^{-1}\frac{\left[E(\text{GeV})\right]^4}{R(\text{meter})}$$

$\simeq$ significant!

For fixed E, proton have a smaller β, γ and much smaller radiation loss

THOMSON SCATTERING



$$\vec{E} = \frac{e}{c^2 R}\,\hat{R}\times(\hat{R}\times\dot{v})\Big|_{ret.} = \frac{e^2}{mc^2 R}\,\hat{R}\times(\hat{R}\times\vec{E_0})\Big|_{ret.}$$

assume: non-relativistic motion ⇨ $\frac{vB}{c} \ll E$,
oscillation small compared wavelength.
applicable when $E_\gamma \gg E_{band}$ but
$E_\gamma \ll m_e c^2$

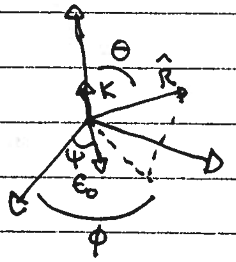
$$m\dot{v} = eE_0 e^{-i\omega t}$$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\left|\hat{R}\times(\hat{R}\times\dot{\vec{v}})\right|^2 = \overbrace{\frac{e^4}{4\pi m^2 c^3}}^{\frac{c}{4\pi}\left(\frac{e^2}{mc^2}\right)^2}\underbrace{\left|\hat{R}\times(\hat{R}\times\vec{E_0})\right|^2}_{\frac{1}{2}\mathrm{Re}\,E_0\cdot E_0^* = \text{average over a cycle}}$$

$$\frac{d\sigma}{d\Omega} = \frac{\text{outgoing radiation / time solidangle}}{\text{incoming energy / time area}} = \frac{c/4\pi\,(e^2/mc^2)^2\,\frac{1}{2}\mathrm{Re}\,E_0 E_0^*\,\sin^2\Theta}{\frac{c}{4\pi}\,\frac{1}{2}\mathrm{Re}\,E_0\cdot E_0^*}$$

$$= \underbrace{\left(\frac{e^2}{mc^2}\right)^2}_{\text{classical electron radius}}\sin^2\Theta$$

Θ: angle between E_0 and $\hat{R}$



$$\vec{k} = k\,\hat{z}$$

$$\hat{\epsilon}_0 = \cos\psi\,\hat{x} + \sin\psi\,\hat{y}$$

$$\hat{R} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\quad\cos\theta\,\hat{z}$$

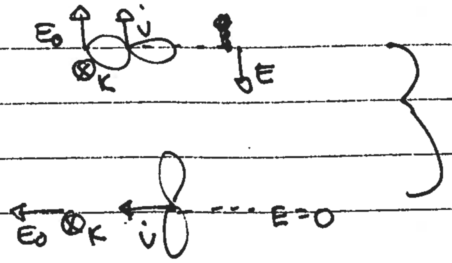
$$\cos\Theta = \hat{\epsilon}_0\cdot\hat{R} = \sin\theta\,(\underbrace{\cos\phi\cos\psi + \sin\phi\sin\psi}_{\cos(\phi-\psi)})$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2\left(1 - \sin^2\theta\cos^2(\phi-\psi)\right),\quad \int\frac{d\psi}{2\pi}\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2\underbrace{\left(1 - \frac{\sin^2\theta}{2}\right)}_{\frac{1+\cos^2\theta}{2}}$$

$$\sigma_T = \int d\Omega\,\frac{d\sigma}{d\Omega} = \int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\,\left(\frac{e^2}{mc^2}\right)^2\frac{1+\cos^2\theta}{2} = \left(\frac{e^2}{mc^2}\right)^2\frac{\cancel{4}\pi}{\cancel{2}}\left(2 + \frac{2}{3}\right)$$

$$= \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)$$

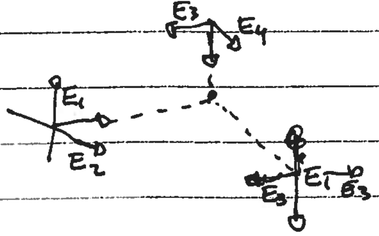
Notice that the scattered wave is polarized:



maximum polarization @ $\Theta = \pi/2$

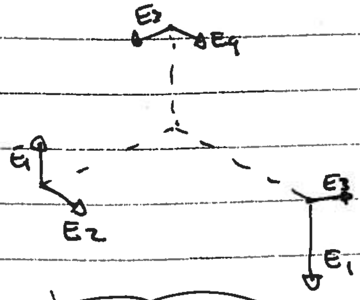
polarization of CMB:

~~the last scattering~~



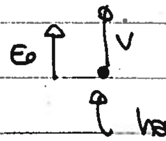
~~Homogeneous~~ isotropic density:
polarization cancels between
scattering from directions
π/2 apart

the last scattering



quadrupole anisotropy in density
⇩
polarization on CMB

RAYLEIGH SCATTERING



harmonically bound particle

(applicable under similar conditions as Thomson scattering but w/ $E_\gamma \ll E_{bound}$)

$$m\ddot{x} + m\omega_0 x = eE_0 e^{-i\omega t}$$

$$m\,x(\omega)\,(-\omega^2+\omega_0^2) = eE_0$$

$$x(\omega) = -\frac{eE_0}{m}\frac{1}{\omega^2-\omega_0^2}$$

$$\dot{x}(\omega) = \frac{ieE_0}{m}\frac{\omega}{\omega^2-\omega_0^2}$$

$$P = \frac{2}{3}\frac{e^2}{c^3}|\ddot{x}|^2 = \underbrace{\frac{2}{3}\frac{e^2}{c^3}\frac{e^2E_0^2}{m^2}}_{\frac{2c}{3}\left(\frac{e^2}{mc^2}\right)^2E_0^2}\frac{\omega^2}{(\omega^2-\omega_0^2)^2}$$

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{2\cancel{c}}{3}\left(\frac{e^2}{mc^2}\right)^2\frac{\cancel{E_0^2}}{\frac{\cancel{c}}{8\pi}\cancel{E_0^2}}}_{\frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2}\frac{\omega^4}{(\omega^2-\omega_0^2)^2}$$

$$\xrightarrow[\omega\ll\omega_0]{} \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2\frac{\omega^4}{\omega_0^4}$$

for transparent gases, $\omega_0 \sim$ UV

more scattering on the bl
less scattering on the sunset

I don't have a clue of how to ulyue

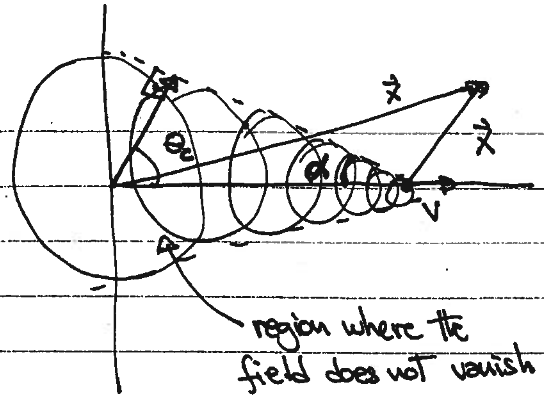
---

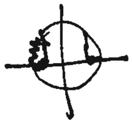
some scales : $\alpha = e^2/\hbar c \simeq 1/137....$ (dimensionless)

$$\underbrace{\frac{\alpha\hbar}{mc}}_{\text{classical electron radius}} \ll \underbrace{\frac{\hbar}{mc}}_{\frac{1}{2\pi}\text{Compton wavelength}} \ll \underbrace{\frac{\hbar}{mc\alpha}}_{\text{Bohr radius}}$$

$\frac{e^2}{mc^2} \simeq 3\times10^{-15}$m $\quad\sim 4\times10^{-13}$m $\quad\simeq 5\times10^{-11}$m

no $\hbar$ hidden in here





$$-\pi < \cos\alpha < 1-\frac{c^2}{n^2v^2}$$

$$\theta_c+\alpha+\frac{\pi}{2}=\pi \Rightarrow \theta_c=\frac{\pi}{2}-\alpha$$

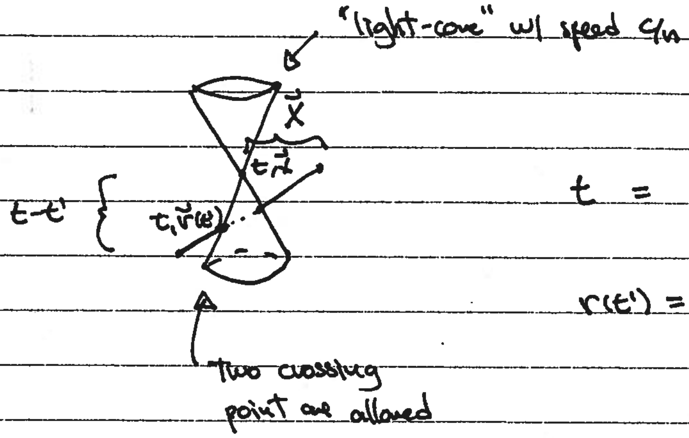
$$\underbrace{\cos\alpha}_{\sqrt{1-\frac{c^2}{n^2v^2}}} = \underbrace{\cos\left(\frac{\pi}{2}-\theta_c\right)}_{\sin\theta_c}$$

$$\cos\theta_c=\sqrt{1-\sin^2\theta_c} = \sqrt{1-1+\frac{c^2}{n^2v^2}} = \frac{c}{nv}$$

A model for $\epsilon(\omega)$ is necessary for a realistic calculation of the radiation. Notice that Čerenkov radiation is part of a larger problem of radiation losses in matter. Maybe next year I'll cover this.

ČERENKOV RADIATION

particle going through medium at $v > \frac{c}{n}$ (assume $\epsilon(\omega) = \epsilon$ for sake of argument)



$$t = \overset{\text{retarded time}}{t'} + \frac{|\vec{x} - \vec{r}(t')|}{c/n}$$

$r(t') = v t'$ (particle trajectory ; $r(0) = 0$ w/o loss of generality)

$$t - t' = \frac{n}{c} \underbrace{|\vec{x} - \vec{v}t'|}_{\vec{X} + v(t-t')}$$

$\vec{X} = \vec{x} - \vec{v}t$ = distance to the present particle position

$$(t-t')^2 - \frac{n^2}{c^2}\vec{X}^2 - \frac{n^2}{c^2}v^2(t-t')^2 - \frac{2n^2}{c^2}\vec{X}\cdot\vec{v}\,(t-t') = 0$$

$$(t-t')^2\left(1 - \frac{n^2v^2}{c^2}\right) - \frac{2n^2}{c^2}\vec{X}\cdot\vec{v}\,(t-t') - \frac{n^2}{c^2}\vec{X}^2 = 0$$

$$t - t' = \frac{\cancel{2}n^2}{c^2}\frac{\vec{X}\cdot\vec{v}}{(1-n^2v^2/c^2)} \pm \sqrt{\frac{\cancel{4}n^4}{c^4}(\vec{X}\cdot\vec{v})^2 + \cancel{4}\left(1 - \frac{n^2v^2}{c^2}\right)\frac{n^2}{c^2}\vec{X}^2}\ \frac{1}{(1-n^2v^2/c^2)}$$

$$= -\frac{\vec{X}\cdot\vec{v}}{v^2 - c^2/n^2} \pm \frac{\sqrt{(\vec{X}\cdot\vec{v})^2 - (v^2 - c^2/n^2)\vec{X}^2}}{v^2 - c^2/n^2}$$

$$= \frac{1}{v^2 - c^2/n^2}\left[-\vec{X}\cdot\vec{v} \pm \sqrt{(\vec{X}\cdot\vec{v})^2 - (v^2 - c^2/n^2)\vec{X}^2}\right]$$

For $v < c/n$, $\sqrt{\ }$ is real, $\sqrt{\ } > |X\cdot v|$ ⇨ one real positive root ⇨ usual case

For $v > c/n$, $\sqrt{\ }$ real ⇨

$\vec{X}\cdot\vec{v} = Xv\cos\alpha$, $\cancel{X^2}v^2\cos^2\alpha > \left(v^2 - \frac{c^2}{n^2}\right)\cancel{X^2} \Rightarrow \cos^2\alpha > 1 - \frac{c^2}{n^2v^2}$

$\vec{X}\cdot\vec{v} < 0$ ⇨ $\cos\alpha < 0$

MULTIPOLE RADIATION

In Lorentz gauge:

$$\phi(r,t) = \int dr' \, \frac{\rho(r', t - |r-r'|/c)}{|r-r'|}$$

$$\vec{A}(r,t) = \frac{1}{c}\int dr' \, \frac{\vec{J}(r', t - |r-r'|/c)}{|r-r'|}$$

For sources varying as $\sim e^{-i\omega t}$ it is convenient to Fourier transform in time

$$\phi(r,t) = \int \frac{d\omega}{2\pi} \, e^{-i\omega t} \, \phi(r,\omega)$$

$$\vdots$$

so

$$\phi(r,\omega) = \int dr' \, \frac{\rho(r',\omega)}{|r-r'|} \, e^{i\overbrace{\frac{\omega}{c}}^{K}|r-r'|}$$

$$\vec{A}(r,t) = \int dr' \, \frac{\vec{J}(r',\omega)}{|r-r'|} \, e^{iK|r-r'|}$$

fields:

$$\boxed{\vec{B} = \nabla\times\vec{A}}, \quad \cancel{\nabla\times\vec{B} = \nabla\underbrace{\nabla\cdot\vec{A}}_{-\frac{1}{c}\frac{\partial\phi}{\partial t}} - \underbrace{\nabla^2\vec{A}}_{\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}} = -\frac{1}{c}\frac{\partial}{\partial t}\Big[\underbrace{-\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}}_{\vec{E}}\Big]}$$

$$\cancel{\frac{1}{c}\frac{\partial}{\partial t}} \; \nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \overset{\swarrow \text{ away from currents}}{=0} \Rightarrow +\frac{i\omega}{c}\vec{E}(\omega) + \nabla\times\vec{B}(\omega) = 0 \Rightarrow \boxed{\vec{E}(\omega) = \frac{i}{K}\nabla\times\vec{B}(}$$

We are interested in the "far" or "radiation" zone:

$$\underset{\substack{\uparrow\\ \text{size of}\\ \text{source}}}{d} \ll \underset{\substack{\uparrow\\ \text{wavelength}\\ 2\pi/k}}{\lambda} \ll \underset{\substack{\uparrow\\ \text{distance}\\ \text{from source to}\\ \text{observation point}}}{r}$$

ELECTRIC DIPOLE APPROXIMATION (1st order in d/λ)

$$\vec{A}(r,\omega) \underset{d\ll r}{\simeq} \frac{1}{c}\int dr'\, \vec{J}(r',\omega)\, \frac{e^{ikr}}{r} = \underset{\text{spherical wave}}{\frac{e^{ikr}}{cr}} \underbrace{\int dr'\, \vec{J}(r',\omega)}$$

$$\underbrace{\int dr'\, \partial_j' (r'^i J^j)}_{=0} = \int \underbrace{\underbrace{\partial_j r'^i}_{\delta^{ij}} J^j}_{J^i} + r'^i \partial_j' J^j$$

$$= -\frac{e^{ikr}}{cr}\int dr'\, \vec{r}'\, \nabla\cdot J(r',\omega)$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot J=0$$

$$-i\omega\rho(\omega)+\nabla\cdot J=0$$

$$= \frac{i\omega}{c}\frac{e^{ikr}}{r}\underbrace{\int dr'\, \vec{r}'\, \rho(r',\omega)}$$

$$\vec{p}(\omega) = \text{electric dipole}$$

$$= ik\,\frac{e^{ikr}}{r}\,\vec{p}(\omega)$$

$$\vec{B} = \nabla\times\vec{A} = \frac{ik}{c}\left[\frac{ik}{r}-\frac{1}{r^2}\right]e^{ikr}\,\hat{r}\times\vec{p}(\omega) \underset{\lambda\ll r}{\simeq} -k^2\,\frac{e^{ikr}}{r}\,\hat{r}\times\vec{p}(\omega)$$

$$\nabla\times(f\vec{v}) = \nabla f\times\vec{v} + f\,\nabla\times\vec{v}$$

$$\vec{E} = \frac{i}{k}\nabla\times\vec{B} = -ik\,\nabla\times\left[\frac{e^{ikr}}{r}\,\hat{r}\times\vec{p}(\omega)\right]$$

$$\nabla\times\left(f(r)\ \vec r\times\vec a\right) = \nabla f\times(\vec r\times\vec a) + f\,\underbrace{\nabla\times(\vec r\times\vec a)}_{\vec r\,\underbrace{\cancel{\nabla\cdot\vec a}}_{0}\ -\ a\,\underbrace{\nabla\cdot\vec r}_{3}\ +\ (a\cdot\nabla)\vec r\ -\ \underbrace{\cancel{(b\cdot\nabla)\vec a}}_{0}}$$

$$= \nabla f\times(\vec r\times\vec a) - 3f\vec a + f\,\underbrace{(a\cdot\nabla)\vec r}_{\underbrace{a^i\partial_i r^j\,\hat e_j}_{a^i\delta_i{}^j\,\hat e_j\ =\ \vec a}}$$

$$= \nabla f\times(\vec r\times\vec a) - 3f\vec a + f\vec a$$

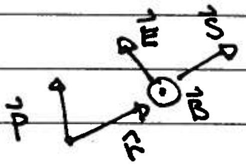
$$= \nabla f\times(\vec r\times\vec a) - 2f\vec a$$

$$\vec E = -ik\ \nabla\times\left[\frac{e^{ikr}}{r^2}\ \vec r\times\vec p\right] = -ik\left[e^{ikr}\left(\frac{ik}{r^2}-\frac{2}{r^3}\right)\hat r\times(\vec r\times\vec p) - 2\,\frac{e^{ikr}}{r^2}\,\vec p\ \right]$$

$$\simeq +k^2\ \frac{e^{ikr}}{r}\ \hat r\times(\hat r\times\vec p) = \vec B\times\hat r$$

$$\langle\vec S\rangle = \frac{c}{4\pi}\,\frac{1}{2}\,\vec E\times\vec B^* = \frac{c}{8\pi}\,(\vec B\times\hat r)\times\vec B^*$$

Time average



$$\left\langle\frac{d\dot P}{d\Omega}\right\rangle = r^2\langle|\vec S|\rangle = r^2\,\frac{c}{8\pi}\,\vec B\cdot\vec B^* = r^2\,\frac{c}{8\pi}\left(+\frac{k^4}{r^2}\right)|p|^2\sin\theta$$

$$= \frac{c}{8\pi}\,k^4\,|p|^2\sin^2\theta$$

$$P = \int d\Omega\,\frac{dP}{d\Omega} = \int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\ \frac{ck^4|p|^2}{8\pi}\,(1-\cos^2\theta) = \frac{ck^4|p|^2}{8\pi}\,2\pi\left(2-\frac{2}{3}\right)$$

$$= \frac{ck^4|p|^2}{3}$$

MAGNETIC DIPOLE AND ELECTRIC QUADRUPOLE APPROXIMATION (2^{nd} order in d/λ)

$$\frac{e^{ik|r-r'|}}{|r-r'|} \simeq \frac{1}{r}\, e^{ikr\sqrt{1-2\frac{r\cdot r'}{r^2}+\frac{r'^2}{r^2}}} \simeq \frac{e^{ikr}}{r}\left(1-\frac{r\cdot r'}{r^2}+\dots\right)$$

$$\simeq \frac{e^{ikr}}{r}\left(1+\underset{\uparrow}{\frac{ik\, r\cdot r'}{r}}+\dots\right)$$

keep this one now

$\sim \mathcal{O}\left(\frac{d}{\lambda}\right)$

$$\tilde{A}(r,\omega) \simeq \frac{1}{c}\int dr'\, \vec{J}(r';\omega)\, \frac{e^{ikr}}{r}\, ik\,\hat{r}\cdot r'$$

only the 2nd order term

$$\simeq \frac{ik}{c}\frac{e^{ikr}}{r}\int dr'\, \underbrace{\hat{r}\cdot r'\, \tilde{J}(r';\omega)}_{\hat{r}^i\,(r'^i J^j)\,\hat{e}_j}$$

$$r'^i J^j = \frac{1}{2}\left(r'^i J^j + r'^j J^i\right)+\frac{1}{2}\underbrace{\left(r'^i J^j - r'^j J^i\right)}_{(\delta^{im}\delta^{jn}-\delta^{in}\delta^{jm})\,r'^m J^n}$$

$$\underbrace{(\delta^{im}\delta^{jn}-\delta^{in}\delta^{jm})\,r'^m J^n}_{\epsilon^{kij}\,\epsilon^{kmn}\,r'^m J^n}$$

$$\underbrace{\epsilon^{kij}\,\epsilon^{kmn}\,r'^m J^n}_{\epsilon^{ijk}\,(r'\times J)^k}$$

$$\simeq \frac{ik}{2c}\frac{e^{ikr}}{r}\int dr'\left[\underset{\uparrow}{(r'\times J)\times\hat{r}} + \hat{r}\cdot r'\,\tilde{J} + \underset{\uparrow}{\tilde{r}'\, r\cdot J}\right]$$

magnetic dipole part

electric quadrupole part

magnetic dipole radiation

$$\vec{A}(r,\omega) = \frac{ik}{\cancel{4\pi}}\frac{e^{ikr}}{r}\underbrace{\frac{1}{2c}\int dr'\left(\vec{r}'\times\vec{J}(r',\omega)\right)\times\hat{r}}_{\vec{m}(\omega)}$$

$$= ik\frac{e^{ikr}}{r}\,\vec{m}\times\hat{r}$$

$$\vec{B} = \nabla\times\vec{A} = -ik\,\nabla\times\left[\frac{e^{ikr}}{r}\,\hat{r}\times\vec{m}\right]$$

$$= k^2\frac{e^{ikr}}{r}\,\hat{r}\times(\hat{r}\times\vec{m})$$

just like $\vec{E}$ from electric dipole radiation

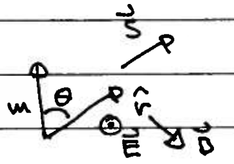
since

$$\nabla\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0 \Rightarrow \nabla\times\vec{E} - ik\vec{B}=0 \Rightarrow \vec{B} = \frac{-i}{k}\nabla\times\vec{E}$$

$$\nabla\times\vec{B} \frac{1}{-c}\frac{\partial\vec{E}}{\partial t}=0 \Rightarrow \nabla\times\vec{B}+ik\vec{E}=0 \Rightarrow \vec{E} = \frac{i}{k}\nabla\times\vec{B}$$

$$\vec{E} = \underbrace{+\,k^2\frac{e^{ikr}}{r}\,\hat{r}\times\vec{m}}_{\text{like }\vec{B}\text{ for electric dipole w/a minus sign}}$$

$$\langle\vec{S}\rangle = \frac{c}{4\pi}\frac{1}{2}\,E\times B^* = \frac{c}{8\pi}\frac{1}{r^2}k^4\,(\hat{r}\times\vec{m})\times(\hat{r}\times(\hat{r}\times\vec{m}^*))$$



$$= \frac{c}{8\pi}\frac{1}{r^2}k^4\,|\vec{m}|^2\sin^2\theta$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = r^2\langle\vec{S}\rangle = \frac{c}{8\pi}k^4\,|\vec{m}|^2\sin^2\theta$$

$$P = \int d\Omega\,\frac{dP}{d\Omega} = \frac{ck^4}{3}|\vec{m}|^2$$

electric quadrupole radiation

$$\vec{A}(r,\omega) = \frac{ik}{2c}\frac{e^{ikr}}{r}\int dr' \left[\hat{r}\cdot r'\,\vec{J} + \vec{r}'\,\hat{r}\cdot\vec{J}\right]$$

$$\partial'_k\left(r'^i r'^j J^k\right) = r'^j J^i + r'^i J^j + r'^i r'^j \underbrace{\nabla\cdot J}_{-i\omega\rho}$$

$$\vec{A}(r,\omega) = -\frac{k\omega}{2c}\frac{e^{ikr}}{r}\hat{r}^i \underbrace{\int dr'\, r'^i r'^j\, \rho(r',\omega)}_{\int dr' \left(r'^i r'^j - \frac{r'^2}{3}\delta^{ij}\right)\rho(r',\omega)}\ \hat{e}_j$$

Q this term doesn't radiate, to make sure I'll multiply it by δ to keep track of it

$$\underbrace{\int dr' \left(r'^i r'^j - \frac{r'^2}{3}\delta^{ij}\right)\rho(r',\omega)}_{\frac{1}{3}Q^{ij}(\omega) = \frac{1}{3}\text{ electric quadrupole moment}}$$

$$= -\frac{k^2}{6c}\frac{e^{ikr}}{r}\hat{r}\cdot\overleftrightarrow{Q}$$

$$\vec{B} = \nabla\times\vec{A} = -\frac{k^2}{6c}\nabla\times\left(\frac{e^{ikr}}{r^2}\vec{r}\right)\cdot\overleftrightarrow{Q} = -\frac{k^2}{6c}\nabla\left(\frac{e^{ikr}}{r^2}\right)\times\vec{r}\cdot\overleftrightarrow{Q}$$

$$= -\frac{k^2}{6c}e^{ikr}\left(\frac{ik}{r^2}\cancel{-\frac{2}{r^3}}^{\to 0}\right)\hat{r}\times\vec{r}\cdot\overleftrightarrow{Q}$$

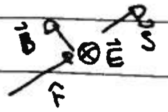
$\nabla\times(f\vec{a}) = \nabla f\times\vec{a} + f\nabla\times\vec{a}$

$\nabla\times\vec{r} = \begin{vmatrix}\partial_x & \partial_y & \partial_z\\ x & y & z\\ \hat{x} & \hat{y} & \hat{z}\end{vmatrix} = 0$

$$= -\frac{ik^3}{6c}\frac{e^{ikr}}{r}\hat{r}\times\left(\hat{r}\cdot\overleftrightarrow{Q}\right)$$

$\hat{r}\times\left(\hat{r}\cdot\overleftrightarrow{Q}\right) = \epsilon^{mkj}\hat{r}^k\hat{r}^i Q^{ij}\hat{e}_m$, the piece in $Q^{ij} \sim \delta^{ij}$ gives $\epsilon^{mki}\hat{r}^k\hat{r}^i\hat{e}_m$
$= 0$

$$\vec{E} \underset{\text{need to work it out}}{=} -\frac{ik^3}{6c}\frac{e^{ikr}}{r}\left(\hat{r}\times(\hat{r}\cdot\overleftrightarrow{Q})\right)\times\hat{r}$$

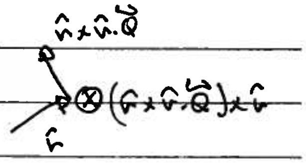


$$\left(\frac{dP}{d\Omega}\right) = r^2 \langle \vec{S} \rangle = r^2 \frac{c}{4\pi}\frac{1}{2}\vec{E}\times\vec{B}^*$$

$$= \frac{c}{8\pi}\frac{k^6}{36}\left[(\hat{r}\times\hat{r}\cdot\overleftrightarrow{Q})\times\hat{r}\right]\times\left[\hat{r}\times\hat{r}\cdot\overleftrightarrow{Q}^*\right]$$

$$= \frac{c\,k^6}{288\pi}\left|(\hat{r}\times\hat{r}\cdot\overleftrightarrow{Q})\times\hat{r}\right|^2$$

$\nwarrow$ complicated angular dependence



$$(\hat{r}\times\hat{r}\cdot\overleftrightarrow{Q})\times\hat{r} = \underset{\epsilon^{nlm}\,\hat{r}^k}{\epsilon^{lkj}}\,\hat{n}^i Q_{ij}\,\hat{r}^m = -(\delta^{km}\delta^{jn} - \delta^{kn}\delta^{jm})\,\hat{r}^k\hat{r}^i\hat{r}^m Q_{ij}$$

$$= -\hat{r}\cdot\overleftrightarrow{Q} + \hat{n}\cdot\overleftrightarrow{Q}\cdot\hat{r}\,\hat{r}$$

$$|(\hat{r}\times\hat{r}\cdot\overleftrightarrow{Q})\times\hat{r}|^2 = (-\hat{r}\cdot\overleftrightarrow{Q} + \hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r})(-\hat{r}\cdot\overleftrightarrow{Q}^* + \hat{r}\cdot\overleftrightarrow{Q}^*\cdot\hat{r})$$

$$= r\cdot\overleftrightarrow{Q}\cdot\overleftrightarrow{Q}^*\cdot\hat{r} + |\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}|^2 -_2 \hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}\ \overset{x}{\hat{r}}\cdot\overleftrightarrow{Q}^*\cdot\hat{r}$$

$$= \hat{r}\cdot\overleftrightarrow{Q}\cdot\overleftrightarrow{Q}^*\cdot\hat{r} - \hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}\ \hat{r}\cdot\overleftrightarrow{Q}^*\cdot\hat{r}$$

$$= \hat{r}_i\, Q_{ik}Q^*_{kj}\,\hat{r}_j - \hat{r}_i\, Q_{ij}\,\hat{r}_j\ \hat{r}_k Q^*_{kl}\,\hat{r}_l$$

$$\int d\Omega\, \hat{r}_i\hat{r}_j = \frac{4\pi}{3}\delta_{ij} \quad \left(\int d\Omega\, \hat{r}_i\hat{r}_i = 4\pi\right)$$

$$\int d\Omega\, \hat{r}_i\hat{r}_j\hat{r}_k\hat{r}_l = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\left(\int d\Omega\, \hat{r}_i\hat{r}_i\hat{r}_k\hat{r}_k = 4\pi = 9 + 3 + 3 = 15\right)$$

24
15
120
24
288
12
360
48 24

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{c k^6}{288\pi c^2}\underbrace{\left[\frac{4\pi}{3}\mathrm{tr}(Q\cdot Q^*) - \frac{4\pi}{15}\left(\underbrace{|\mathrm{tr}\,Q|^2}_{0} + \mathrm{tr}\,QQ^* + \mathrm{tr}\,QQ^*\right)\right]}_{\left(\frac{4\pi}{3} - \frac{8\pi}{15}\right)\mathrm{tr}(QQ^*) = \frac{20\pi - 8\pi}{15}\mathrm{tr}(QQ^*) = \frac{12\pi}{15}\mathrm{tr}(QQ^*)}$$

$$= \frac{c K^6}{\underbrace{15\cdot 24}_{360}}\,\mathrm{tr}(QQ^*)$$