MAGNETOSTATICS

steady currents only
$$\overset{0}{\cancel{\frac{1}{c}\frac{\partial\rho}{\partial t}}} + \nabla\cdot\vec{J} = 0 \Rightarrow \nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}$$

$$\nabla\cdot\vec{B} = 0 \;\downarrow\; \text{no monopoles} \quad \vec{B} = \nabla\times\vec{A}$$

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} \;\downarrow\; \oint d\vec{l}\cdot\vec{B} = \frac{4\pi}{c} I$$ (Ampere's law)

$$\underbrace{\nabla\times\nabla\times\vec{A}}_{} = \frac{4\pi J}{c} \Rightarrow \underbrace{\nabla\nabla\cdot\vec{A}}_{\substack{=0 \text{ in}\\ \text{Coulomb or radiation}\\ \text{gauge}}} - \nabla^2\vec{A} = \frac{4\pi J}{c}$$

$$\epsilon^{mlk}\partial^l \epsilon^{kji}\partial^j A^i \hat{e}_m$$
$$= (\delta^{mj}\delta^{li} - \delta^{lj}\delta^{im})\partial^l\partial^j A^i \hat{e}_m$$
$$= \nabla\nabla\cdot\vec{A} - \nabla^2\vec{A}$$

gauge transformation: $\vec{A} \to \vec{A} + \nabla\chi$, $\vec{B} = \nabla\times\vec{A} \Rightarrow \nabla\times\vec{A} + \underbrace{\nabla\times\nabla\chi}_{=0}$

suppose $\nabla\cdot\vec{A} = f \Rightarrow \nabla\cdot(\vec{A}+\nabla\chi) = \nabla\cdot A + \nabla^2\chi = 0 \Rightarrow \nabla^2\chi = -f$
(Poisson eq.)

$$\nabla^2\vec{A} = -\frac{4\pi}{c}J$$ (Poisson eq. for each component)

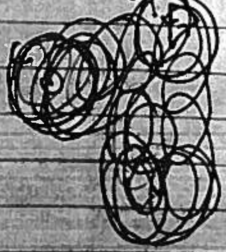
⇓

w/ Trivial b.c.'s
$$\vec{A}(\vec{r}) = \frac{1}{c}\int dr' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

and

$$\vec{B}(\vec{r}) = \nabla\times\frac{1}{c}\int dr' \frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|} = \frac{1}{c}\int dr' \,\nabla_r\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\times\vec{J}(r')$$

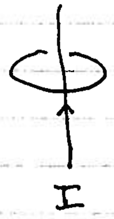
$$\nabla\times(f\vec{V}) = \epsilon_{kji}\partial_j\left(f(r)V_i(r)\right)\hat{e}_k$$
$$= \epsilon_{kji}\,\partial_j f(r) V_i(r)\hat{e}_k$$
$$= \nabla f\times\vec{V}$$



$$= \frac{1}{c}\int dr' \frac{-(\vec{r}-\vec{r}')\times\vec{J}(r')}{|\vec{r}-\vec{r}'|^3}$$ (Biot-Savart law)

$$= \frac{1}{c}\int dr' \,J(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

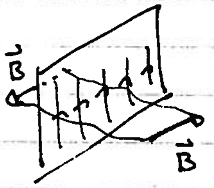
EXAMPLE(S): simple problems can be dealt with Ampere's law



$2\pi r\, B = \frac{4\pi}{c} I \Rightarrow \vec{B} = \frac{2I}{cr}\hat{\phi}$

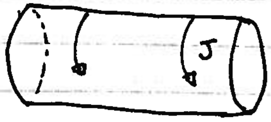
$2\pi r\, B = \frac{4\pi}{c} I \frac{r^2}{R^2} \Rightarrow \vec{B} = \frac{2I}{c}\frac{r}{R^2}\hat{\phi} \quad , \quad r<R$

$\frac{2I}{cr}\hat{\phi} \quad , \quad r>R$



$2L\, B = \frac{4\pi}{c} L J \Rightarrow B = \frac{2\pi}{c} J$

current per length



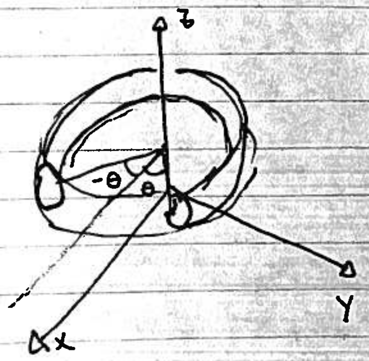
$B_\phi = 0$ because no current through the loop

$B_\rho = 0$ because ~~it should flip sign if J → -J (or solenoid direction flipped)~~ no $\vec{B}$ flow out of  case slice gaussian surface

outside:  $B(r)$ ind. of r $\Rightarrow$ $B=0$

inside:  $L\,B(r) = I N L \frac{4\pi}{c} \Rightarrow B(r) = \frac{4\pi I N}{c}$

turns/length



on the xz plane:

$r' = (\rho'\cos\theta', \rho'\sin\theta', z') \quad , \quad r = (x, 0, z)$

$r - r' = (x - \rho'\cos\theta', -\rho'\sin\theta', z - z')$

$I = I_\rho \hat{\rho} + I_z \hat{z} = (I_\rho \cos\theta', I_\rho \sin\theta', I_z)$

no ε component

$I \times (r - r') = \ldots \sin\theta' \hat{z} + \ldots \sin\theta' \hat{x} + \ldots \hat{y} \sim \hat{y}$

cancels between θ' and -θ'

~~outside: $2\pi r B = 0$, $B = 0$~~

inside: $2\pi r B = \frac{4\pi}{c} I N \Rightarrow B = \frac{2IN}{cr}\hat{\theta}$

total # of winds

Multipole expansion

$$\vec{A}(\vec{r}) = \frac{1}{c}\int dr' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{c}\int dr' \vec{J}(\vec{r}')\left[\frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \dots\right]$$

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r} + \frac{r\cdot r'}{r^3} + \dots$$

$$= \frac{1}{cr}\int dr' \vec{J}(\vec{r}') + \frac{r^i}{cr^3}\int dr' \vec{J}(\vec{r}')\, r'^i$$

But $\nabla\cdot(fg\vec{J}) = g\nabla f\cdot\vec{J} + f\nabla g\cdot\vec{J} + fg\underbrace{\nabla\cdot\vec{J}}_{=0} \Rightarrow \int_V dr'\left[g\nabla f\cdot\vec{J} + f\nabla g\cdot\vec{J}\right] = \underbrace{\oint_{\partial V} da\,\hat{n}\cdot(fg\vec{J})}_{=0}$

because $\vec{J}$ vanishes @ ∂V

with $f=1,\ g=r'^j$, $\int dr'\left[\underbrace{\partial_{r'^i} r'^j}_{\delta^{ij}} J^i\right] = \int dr' J^j = 0$

with $f=r'^i,\ g=r'^j$, $\int dr'\left[r'^j \underbrace{\partial_{r'^k} r'^i}_{\delta^{ki}} J^k + r'^i \underbrace{\partial_{r'^k} r'^j}_{\delta^{kj}} J^k\right] = \int dr'\left[r'^j J^i + r'^i J^j\right] = 0$

$$A^j(\vec{r}) \simeq \frac{1}{cr}\,0 + \frac{r^i}{cr^3}\int dr'\left[r'^i J^j(r')\right] = \frac{1}{cr^3}\int dr' \frac{1}{2}\underbrace{\left[r^i r'^i J^j(r') - r^i r'^j J^i(r')\right]}_{r^m r'^k J^l \underbrace{\left(\delta^{mk}\delta^{lj} - \delta^{ml}\delta^{kj}\right)}_{\epsilon^{nmj}\,\epsilon^{nkl}}}$$

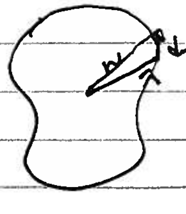
$+-$

$$= \frac{1}{2cr^3}\int dr' \quad (\vec{r}'\times\vec{J})\times\vec{r}$$

$$= \frac{1}{r^3}\underbrace{\left(\frac{1}{2c}\int dr'\ r'\times\vec{J}\right)}_{\vec{m}}\times\vec{r} = \vec{m}\times\frac{\hat{r}}{r^2}$$

(magnetic dipole)

EXAMPLE: loop on a plane



$$m = \frac{I}{2c}\int \vec{r}'\times d\vec{l} = \frac{1}{c} I \times \text{area}$$

bund : $\equiv$: bunch of classical particles and the same q/m

$= \frac{1}{2}\sum_{n=1}^{N} q_n\, \vec{r}_n \times \vec{v}_n = \sum \frac{q_n m_n}{2 m\, m_n}\, \vec{r}_n \times \vec{v}_n = \frac{q}{2m} \underbrace{\sum_{n=1}^{N} m_n\, \vec{r}_n \times \vec{v}_n}_{\vec{L}}$

: (It fails for e^- by a factor of $\simeq 2$!)

$= \sum_{i=1}^{N} \vec{L}$

:

Magnetism in matter

In presence of magnetized material:

$$\vec{A}(\vec{r}) = \frac{1}{c}\int dr' \left[\frac{J(\vec{r}')}{|r-r'|} + c\, M(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|r-r'|^3} \right]$$

not really allowed to use dipole approximation here but, just like in dielectrics, the result is right (see below)

$$= \frac{1}{c}\int dr' \left[\frac{\vec{J}(\vec{r}')}{|r-r'|} + c\,\vec{M}(\vec{r}') \times \nabla' \left(\frac{1}{|r-r'|} \right) \right]$$

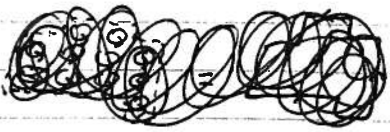
$\nabla \times (fV) = \nabla f \times V + f \nabla \times V$

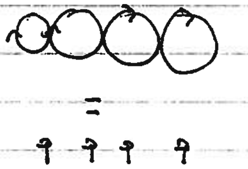
$\Downarrow$

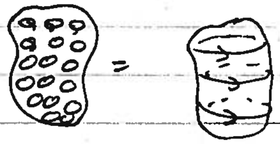
$V \times \nabla f = f \nabla \times V - \nabla \times (fV)$

$$= \frac{1}{c}\int dr' \left[\frac{\underbrace{\vec{J}(\vec{r}')}_{\text{“free current”}} + \underbrace{c\nabla' \times \vec{M}(\vec{r}')}_{\text{“magnetization current”}}}{|\vec{r}-\vec{r}'|} \right] \underbrace{- \frac{c}{c}\int dr' \nabla \times \left(\frac{M(r')}{|r-r'|} \right)}_{-\frac{c}{c}\oint da' \,\hat{n} \times \frac{M(r')}{|r-r'|}}$$

“superficial magnetization current”







$$\nabla \times \vec{B} = \frac{4\pi}{c}(\vec{J}_f + \vec{J}_m) = \frac{4\pi}{c}(\vec{J} + c\nabla \times \vec{M})$$

$$\Downarrow$$

$$\nabla \times \underbrace{(\vec{B} - 4\pi\vec{M})}_{\equiv \vec{H}} = \frac{4\pi}{c}\vec{J}_f$$

linear, isotropic, local materials:

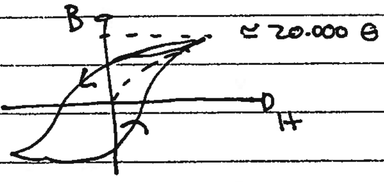
$\vec{B} = \mu \vec{H}$ (analogue of μ is $1/\epsilon$!)

diamagnetic: $\mu < 1$, very common, $1-\mu \simeq 10^{-5}$

paramagnetic: $\mu > 1$ (made of permanent dipoles),

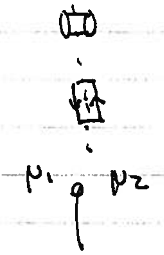
ferromagnets: M in the absence of H (non-linear, true non-locality (histeresis))

iron, nickel and cobalt mainly



superconductors: $\mu = 0$, B fields expelled

boundary conditions $\nabla\cdot B=0$, $\nabla\times H=\frac{4\pi}{c}J_s$



assumes no superficial free currents

$$\underbrace{B_\perp^1}_{\mu_1 H_\perp^1} = \underbrace{B_\perp^2}_{\mu_2 H_\perp^2}$$

$$\underbrace{H_\parallel^1}_{\frac{1}{\mu_1}B_\parallel^1} = \underbrace{H_\parallel^2}_{\frac{1}{\mu_2}B_\parallel^2}$$

analogy w/ electrostatics : in the absence of currents or charges

$$\begin{matrix}\nabla\cdot D=0 \\ \nabla\times E=0 \\ D=\epsilon E = E+4\pi P\end{matrix} \quad \Longleftrightarrow \quad \begin{matrix}\nabla\cdot B=0 \\ \nabla\times H=0 \\ B=\mu H = H+4\pi M\end{matrix}$$

$\Downarrow$

$D \leftrightarrow B$

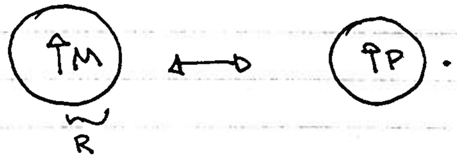
$E \leftrightarrow H$

$\epsilon \leftrightarrow \mu$

$P \leftrightarrow M$

$\phi \leftrightarrow \phi_M$, $H=-\nabla\phi_M$

EXAMPLE: uniformly magnetized sphere



$E_{in} = -\frac{4\pi}{3}P$

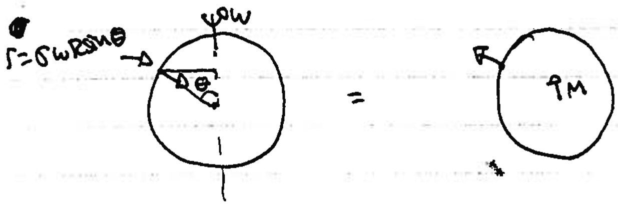
$E_{out} = \frac{4\pi R^3}{3}P\,\frac{\cos\theta}{r^2}$ (electric dipole @ center)

$H_{in} = -\frac{4\pi}{3}M$

$B_{in} = -\frac{4\pi}{3}M + 4\pi M = \frac{8\pi}{3}M$

$H_{out} = \frac{4\pi R^3}{3}M\,\frac{\cos\theta}{r^2}$ (magnetic dipole @ center)

EXAMPLE: rotating charged sphericalshell



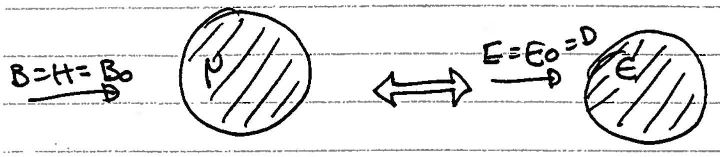
$J = c\,\hat{n}\times\vec{M} = cM\sin\theta$ ⟹ same as before with $M=\frac{\sigma\omega R}{c}$

~~check units : $\left[\frac{\sigma\omega R}{c}\right] = [\sigma] = \left[\frac{\text{charge}}{L^2}\right]$~~

~~$[M] = [B] = \left[\frac{1}{c}J\right] = $ [illegible]~~

$\nabla\times B = \frac{4\pi}{c}$ [illegible] $[B]$

EXAMPLE : magnetic material sphere in external field

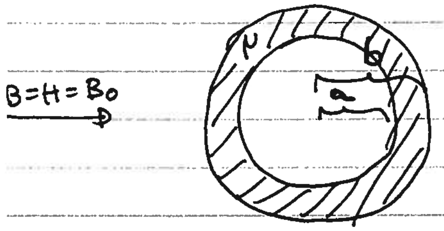


$$H_{in} = \frac{3}{2+\mu} B_0$$

$$B_{in} = \mu H_{in} = \frac{3\mu}{2+\mu} B_0$$

$$M = \frac{B-H}{4\pi} = \frac{B_0}{4\pi}\underbrace{\left(\frac{3\mu}{2+\mu} - \frac{3}{2+\mu}\right)}_{3\frac{(\mu-1)}{2+\mu}} = \frac{3B_0}{4\pi}\left(\frac{\mu-1}{\mu+2}\right)$$

EXAMPLE : spherical shell of magnetic material; magnetic shielding



$$r>b: \quad \phi_M = \sum_{l=0}^{\infty} \frac{A_l}{r^{l+1}} P_l(\cos\theta) - B_0 r \overbrace{\cos\theta}^{P_1(\cos\theta)}$$

$$a<r<b: \quad \phi_M = \sum_{l=0}^{\infty}\left[B_l r^l + \frac{C_l}{r^{l+1}}\right] P_l(\cos\theta)$$

$$r<a: \quad \phi_M = \sum_{l=0}^{\infty} D_l r^l P_l(\cos\theta)$$

$$\left.\frac{\partial}{\partial\theta}\phi_M\right|_{r=b} \Rightarrow \sum_l \frac{A_l}{b^{l+1}} P_l' - B_0 b \underset{\delta_{l1}}{P_1'} = \sum_l \left(B_l b^l + \frac{C_l}{b^{l+1}}\right) P_l'$$

$$\left.\frac{\partial}{\partial r}\phi_M\right|_{r=b} \Rightarrow \sum_l -(l+1)\frac{A_l}{b^{l+2}} P_l - B_0 P_1 \delta_{l1} = \mu \sum_l \left(l B_l b^{l-1} - (l+1)\frac{C_l}{b^{l+2}}\right) P_l$$

$$\Downarrow$$

$$l \neq 1 \quad \frac{A_l}{b^{2l+1}} = B_l + \frac{C_l}{b^{2l+1}}, \quad \frac{-A_l(l+1)}{b^{2l+1}} = \frac{\mu l B_l}{l+1} - \frac{C_l(l+1)}{b^{2l+1}} \Rightarrow B_l = C_l = 0, \; l \neq 1$$

$$\boxed{\frac{A_1}{b^2} - B_0 b = B_1 b + \frac{C_1}{b^2}} \quad \boxed{-2\frac{A_1}{b^3} - B_0 = \mu B_1 - 2\mu \frac{C_1}{b^3}}$$

$$\left.\frac{\partial}{\partial\theta}\phi_M\right|_{r=a} \Rightarrow \boxed{B_1 a + \frac{C_1}{a^2} = D_1 a}$$

$$\left.\frac{\partial}{\partial r}\phi_M\right|_{r=a} \Rightarrow \boxed{\mu\left(B_1 - 2\frac{C_1}{a^3}\right) = D_1}$$

i) $A_1 - b^3 B_1 - C_1 = b^3 B_0$

ii) $2A_1 + \mu b^3 B_1 - 2\mu C_1 = -b^3 B_0$

iii) $a^3 B_1 + C_1 - a^3 D_1 = 0$

iv) $\mu a^3 B_1 - 2\mu C_1 - a^3 D_1 = 0$

ii) $- \mu$ i) $=$ $(2-\mu)A_1 - 3\mu C_1 = -b^3 B_0(+1+\mu)$

iv) $- \mu$ iii) $=$ $-3\mu C_1 - a^3 D_1 (\mu - 1) = 0$

[illegible]

iii) + iv) $\begin{pmatrix} a^3 & 1 \\ \mu a^3 & -2\mu \end{pmatrix}\begin{pmatrix} B_1 \\ C_1 \end{pmatrix} = +a^3 \begin{pmatrix} P_1 \\ D_1 \end{pmatrix}$

$$\underbrace{\underbrace{\frac{1}{-2\mu a^3 - \mu a^3}\begin{pmatrix} -2\mu & -1 \\ -\mu a^3 & a^3 \end{pmatrix}}_{\frac{1}{3\mu a^3}\begin{pmatrix} 2\mu & 1 \\ \mu a^3 & -a^3 \end{pmatrix}}\begin{pmatrix} a^3 & 1 \\ \mu a^3 & -2\mu \end{pmatrix}\begin{pmatrix} B_1 \\ C_1 \end{pmatrix}}_{\begin{pmatrix} B_1 \\ C_1 \end{pmatrix}} = \frac{1}{3\mu a^3}\begin{pmatrix} 2\mu & 1 \\ \mu a^3 & -a^3 \end{pmatrix} a^3 \begin{pmatrix} P_1 \\ D_1 \end{pmatrix} = \frac{1}{3\mu}\begin{pmatrix} (2\mu+1)D_1 \\ (\mu-1)a^3 D_1 \end{pmatrix}$$

$B_1 = \frac{2\mu+1}{3\mu} D_1$

$C_1 = \frac{\mu-1}{3\mu} a^3 D_1$

v) i) $\Rightarrow A_1 \underbrace{- b^3 \frac{2\mu+1}{3\mu} D_1 - \frac{(\mu-1)}{3\mu} a^3 D_1}_{-\frac{D_1}{3\mu}((2\mu+1)b^3 - (\mu-1)a^3)} = b^3 B_0$

vi) ii) $\Rightarrow 2A_1 + \underbrace{\mu b^3 \frac{2\mu+1}{3\mu} D_1 - 2\mu \frac{(\mu-1)}{3\mu} a^3 D_1}_{\frac{D_1}{3\mu}(\mu(2\mu+1)b^3 - \mu 2 a^3 (\mu-1))} = -b^3 B_0$

v) + vi) $3A_1 + \frac{D_1}{3}\left[\underbrace{(2\mu+1)b^3 - \frac{2\mu+1}{\mu} b^3 - 2(\mu+1)a^3 + \frac{(\mu+1)}{\mu} a^3}_{b^3(2\mu+1-2-\frac{1}{\mu}) + a^3(-\mu-1+1+\frac{1}{\mu})}\right] = 0$

$= b^3(2\mu - 1 - \frac{1}{\mu}) + a^3(\frac{1}{\mu} - \mu)$

v) -2 v) $\frac{D_1}{3\mu}\Big(\underbrace{\mu(2\mu+1)b^3 - 2(\mu-1)a^3 + \frac{2}{\mu}(2\mu+1)b^3 - \frac{2}{\mu}(\mu-1)a^3}_{b^3(2\mu^2+\mu+4\mu+\frac{2}{\mu}) + a^3(-2\mu+2+\underset{-2\mu}{}+\frac{2}{\mu})}\Big) = -3b^3 B_0$

$\frac{D_1}{3}\left[b^3(2\mu - 3 - \frac{2}{\mu}) + 2a^3(\frac{1}{\mu} - \mu)\right] = -3b^3 B_0 \Rightarrow D_1 = \frac{-9b^3 B_0}{b^3(2\mu - 3 - \frac{2}{\mu}) + 2a^3(\frac{1}{\mu} - \mu)}$

v): i) $\Rightarrow$ $$A_1 - b^3 \frac{2\mu+1}{3\mu} D_1 - \frac{(\mu-1)}{3\mu} a^3 D_1 = b^3 B_0$$

$$A_1 + \frac{D_1}{3\mu}\left[-(2\mu+1)\, b^3 - (\mu-1)a^3\right]$$

vi): ii) $\Rightarrow$ $$2A_1 + \underbrace{\mu b^3 \frac{2\mu+1}{3\mu} D_1 - 2\mu \frac{(\mu-1)}{3\mu} a^3 D_1}_{\frac{D_1}{3\mu}\left[\mu(2\mu+1)\, b^3 - 2\mu(\mu-1)a^3\right]} = -b^3 B_0$$

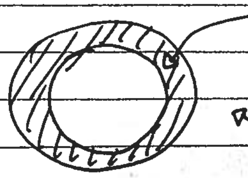
vi) -2 v) $\Rightarrow$ $$\frac{D_1}{3\mu}\underbrace{\left[\mu(2\mu+1)b^3 - 2\mu(\mu-1)a^3 + 2(2\mu+1)b^3 + 2(\mu-1)a^3\right]}_{\underbrace{b^3\left(2\mu^2+\mu+4\mu^2+2\right) + a^3\left(-2\mu^2+2\mu+2\mu-2\right)}_{b^3(2\mu+1)(\mu+2) \quad - a^3\, 2(\mu-1)^2}} = -3b^3 B_0$$

$$D_1 = -\frac{9\mu\, b^3}{b^3(2\mu+1)(\mu+2) - 2a^3(\mu-1)^2} B_0$$

$$D_1 \xrightarrow[a\to b]{} -\frac{9\mu}{\cancel{2\mu^2}+5\mu+\cancel{2}-\cancel{2\mu^2}\cancel{-2}+4\mu} B_0 = -\frac{9\mu}{9\mu} B_0 = -B_0$$

$$\phi_M(r<a) = -B_0 r = \text{external field}$$

$$D_1 \xrightarrow[\mu\to\infty]{} -\frac{9\mu\, b^3 B_0}{b^3 2\mu^2 - 2a^3\mu^2} \to 0 \quad \text{(perfect shielding)}$$



$\mu H_\perp^1 = H_\perp^2 \xRightarrow[\mu\to\infty]{} H_\perp^1 = 0$ (like $E_\perp$ in a conductor)