Note: In the following solutions to the homework, you may find errors. In some cases they may be minor typos and in other cases, the errors could be more severe. If you believe a solution is wrong, discuss it with your peers. If you still believe a solution is wrong, talk to the grader during his office hours (Justin Wilson, PHYS 4219, T 2:00-3:30 or F 10:00-11:30), so that the solution can be fixed. On the other hand, if it is just a typo, send an e-mail to the grader at: `jwilson.thequark@gmail.com`.

Notation/Convention:

- Gaussian units with $c \neq 1$ are used throughout. (Let the grader know if this is violated, so that it can be corrected.)
- The metric used is as follows:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

- The coordinate $x^0 = ct$.
- For derivatives $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$ and $\partial_j = \frac{\partial}{\partial x^j}$. Care needs to be taken since $\partial^j = -\partial_j$ with the metric we are using.
- Summation over repeated Lorentz indices is done when one is up and the other is down. Summation over Cartesian coordinates is done when two are repeated no matter their location.

Problem 9.1 Uniformly Moving Charge

On the first homework you calculated the field due to a charge moving with constant velocity by taking the Coulomb field and boosting it. Use the retarded fields computed in class to arrive at the same result. Does the electric field point towards the present position of the charge or to the position previously occupied by the charge at the “retarded time” $t = R/c$?

Solution

See section 14.1 of Jackson for this derivation.

Problem 9.2 Bremsstrahlung

A *nonrelativisitic* particle of charge Ze, mass m and kinetic energy E makes a head-on collision with a fixed central force field of finite range. The interaction is repulsive and described by a potential $V(r)$ that becomes larger than E at some finite distance r_{min}.

a. The particle, stopped by the force field, radiates. Show that the total power radiated is given by

$$W = \frac{4}{3}\frac{Z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\text{min}}}^{\infty}\left|\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right|^2 \frac{\mathrm{d}r}{\sqrt{V(r_{\text{min}})-V(r)}}$$

b. If the potential has the Coulomb form $V(r) = Z'Ze^2/r$ show that the total power radiated is

$$W = \frac{8}{45}\frac{Zmv_0^5}{Z'c^3}.$$

Solution

Part a.

Looking at Eq. (14.21) in Jackson, we get the power radiated per solid angle for a charge:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{Z^2e^2}{4\pi c^3}\left|\dot{\mathbf{v}}\right|^2 \sin^2\Theta,$$

but we know from classical mechanics that

$$m|\dot{\mathbf{v}}| = |\boldsymbol{\nabla} V|\,,$$

but since this is a head on collision on a central force field, the motion is confined radially, so we can replace $\boldsymbol{\nabla}$ with $\mathrm{d}/\mathrm{d}r$. If we now use this to find the total power radiated

$$\begin{aligned} P &= \int \frac{\mathrm{d}P}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \frac{Z^2e^2}{4\pi c^3 m^2}\left|\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right|^2\int_0^{\pi}\sin^2\Theta\,\mathrm{d}\Theta\int_0^{2\pi}\mathrm{d}\phi \\ &= \frac{2}{3}\frac{Z^2e^2}{c^3m^2}\left|\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right|^2. \end{aligned}$$

Now the particle follows a path $r(t)$, and thus, for the total power we need to integrate over all time:

$$W = \int P(t)\,\mathrm{d}t = \frac{2}{3}\frac{Z^2e^2}{c^3m^2}\int_{-\infty}^{\infty}\mathrm{d}t\left|\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right|^2_{r=r(t)}.$$

First of all, we have chosen our zero in time such that $r(0) = r_{\text{min}}$, and so the integrals from 0 to ∞ is the same as the integral from $-\infty$ to 0. Thus, we have

$$W = \frac{4}{3}\frac{Z^2e^2}{c^3m^2}\int_0^{\infty}\mathrm{d}t\left|\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right|^2_{r=r(t)}.$$

At this point, we use conservation of energy which at $r = r_{\text{min}}$ we have no kinetic energy, giving us

$$V(r_{\text{min}}) = \tfrac{1}{2} m\dot{r}^2 + V(r),$$

and solving this we get

$$\mathrm{d}t = \frac{\mathrm{d}r}{\sqrt{\frac{2}{m}(V(r_{\text{min}}) - V(r))}}.$$

Using this to change variables of integration we have $r(0) = r_{\text{min}}$ and $r(\infty) = \infty$, thus

$$W = \frac{4}{3}\frac{Z^2e^2}{c^3m^2}\sqrt{\frac{m}{2}}\int_{r_{\text{min}}}^{\infty}\left|\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right|^2 \frac{\mathrm{d}r}{\sqrt{V(r_{\text{min}}) - V(r)}}.$$

Part b.

First, note that by conservation of energy $V(r_{\text{min}}) = \frac{1}{2}mv_0^2$ where v_0 is the velocity at infinity (where the potential is zero). Now, we noting the r dependence of V, we have

$$\frac{\mathrm{d}V(r)}{\mathrm{d}r} = -\frac{V(r)}{r}.$$

If we were to change variables to $u = \frac{1}{2}mv_0^2 - V(r)$ now, we have

$$\left|\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right|^2 = \frac{(\frac{1}{2}mv_0^2 - u)^2}{r^2} \qquad\qquad \mathrm{d}r = \frac{r\,\mathrm{d}u}{V} = \frac{r^2\,\mathrm{d}u}{Z'Ze^2},$$

and the integral becomes (letting $A = \frac{1}{2}mv_0^2$)

$$\begin{aligned}
W &= \frac{4}{3}\frac{Z^2e^2}{c^3m^2}\sqrt{\frac{m}{2}}\frac{1}{Z'Ze^2}\int_0^A \frac{(A-u)^2}{\sqrt{u}}\mathrm{d}u \\
&= \frac{4}{3}\frac{Z}{Z'c^3m^2}\sqrt{\frac{m}{2}}\int_0^A \left[A^2u^{-1/2} - 2Au^{1/2} + u^{3/2}\right]\mathrm{d}u \\
&= \frac{4}{3}\frac{Z}{Z'c^3m^2}\sqrt{\frac{m}{2}}A^{1/2}\left[2A^2 - \tfrac{4}{3}A^2 + \tfrac{2}{5}A^2\right] \\
&= \frac{8}{45}\frac{Zmv_0^5}{Z'c^3}.
\end{aligned}$$

Precisely the result we were looking for.