ELECTROSTATICS

no time dependence, no currents $\Rightarrow$

$$\nabla\cdot\vec{E} = 4\pi\rho \iff \oint_{\partial V} da\, \hat{n}\cdot\vec{E} = 4\pi\int d^3\vec{r}\,\rho = 4\pi Q \quad \text{(Gauss law)}$$

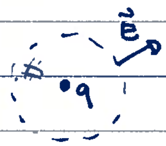
$$\nabla\times\vec{E} = 0 \iff \oint d\vec{\ell}\times\vec{E} = 0$$

$$\Downarrow$$

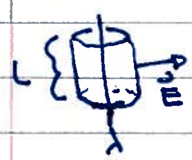
$$\vec{E} = -\nabla\underbrace{\phi}_{A^0} \Rightarrow \nabla^2\phi = -4\pi\rho \quad \text{(Poisson eq.)}$$

PROBLEM 1. : given a $\rho(\vec{r})$, find $\phi(\vec{r})$ (and $\vec{E}(\vec{r})$), assuming $E(r\to\infty)\to 0$

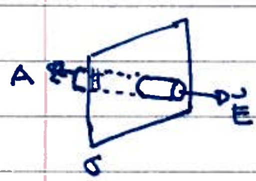
- Symmetry and Gauss law take care of elementary cases



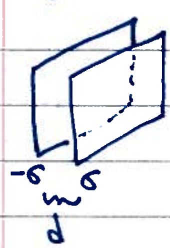
$4\pi r^2 E = 4\pi q \Rightarrow \vec{E} = \frac{q}{r^2}\hat{r}$



$L\, 2\pi r\, E = 4\pi L\lambda \Rightarrow E = \frac{2\lambda}{r}\hat{r}$

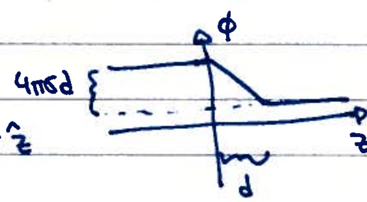


$2EA = 4\pi A\sigma \Rightarrow \vec{E} = 2\pi\sigma\hat{z}$ or $-2\pi\sigma\hat{z}$

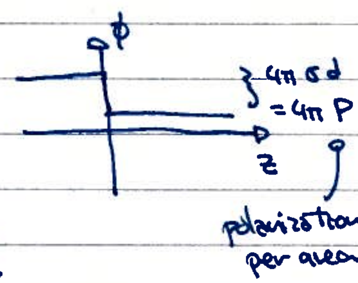


outside : $\vec{E} = 0$

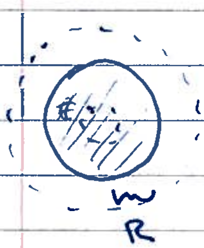
inside : $\vec{E} = -4\pi\sigma\hat{z}$



$d\to 0$



$\vec{P} = -\vec{E}$



outside : $4\pi r^2 E = 4\pi Q \Rightarrow \vec{E} = \frac{Q}{r^2}\hat{r}$

inside : $4\pi r^2 E = 4\pi Q\frac{r^3}{R^3} \Rightarrow \vec{E} = \frac{Qr}{R^3}\hat{r}$

- direct integration of Poisson (or Coulomb) law

$$\nabla^2 \phi(\vec{r}) = -4\pi \rho(\vec{r})$$

linear operator (∇^2), desired (ϕ), given (ρ)

$$A_{ij} v_j = w_i$$

linear operator (A_{ij}), desired (v_j), given (w_i)

MULTIPLY BOTH SIDES BY A^{-1}

$$\underbrace{(A^{-1})_{ki} A_{ij}}_{\delta_{kj}} v_j = (A^{-1})_{ki} w_i$$

$$\Downarrow$$

$$v_k = (A^{-1})_{ki} w_i$$

SAME A^{-1} FOR ALL w's

$$\underbrace{\int d\vec{r}'\, \delta(\vec{r}-\vec{r}')\, \nabla^2_{r'} \phi(\vec{r}')}_{\int d\vec{r}'\, \nabla^2_{r'} \delta(\vec{r}-\vec{r}')\, \phi(\vec{r}')} = -4\pi \rho(\vec{r})$$

MULTIPLY BOTH SIDES BY $G(\vec{r}'', \vec{r})$, THE INVERSE OF $\int dr'\, \nabla^2_{r'} \delta(r-r')$

$$\underbrace{\int d\vec{r}\, G(\vec{r}'', \vec{r}) \int d\vec{r}'\, \underbrace{\nabla^2_{r'} \delta(\vec{r}-\vec{r}')}_{\nabla^2_{r} \delta(\vec{r}-\vec{r}')} \phi(\vec{r}')}_{\int d\vec{r}' \underbrace{\int d\vec{r}\, G(\vec{r}'', \vec{r})\, \nabla^2_{r} \delta(\vec{r}-\vec{r}')}_{\nabla^2_{r'} G(\vec{r}'', \vec{r}') \equiv -4\pi \delta(\vec{r}''-\vec{r}')}} = -\int d\vec{r}\, G(\vec{r}'', \vec{r})\, 4\pi \rho(\vec{r})$$

usual convention

$$\Downarrow$$

$$\boxed{\begin{array}{l} \phi(\vec{r}) = \int d\vec{r}'\, G(\vec{r}, \vec{r}')\, \rho(\vec{r}') \\ \text{with} \quad \nabla^2_{r'} G(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r}-\vec{r}') \end{array}}$$

How to find $G(\vec{r},\vec{r}')$?

$$\underbrace{\nabla^2_{r}}_{\sim 1/L^2}\ \underbrace{G(\vec{r},\vec{r}')}_{\sim 1/L} = -4\pi\ \underbrace{\delta(\vec{r}-\vec{r}')}_{\sim 1/L^3}$$

$$G(\vec{r},\vec{r}') \underset{\text{Translation symmetry}}{=} G(\vec{r}-\vec{r}') \underset{\text{rotational invariance}}{=} G(|\vec{r}-\vec{r}'|) = \frac{A}{|\vec{r}-\vec{r}'|}$$

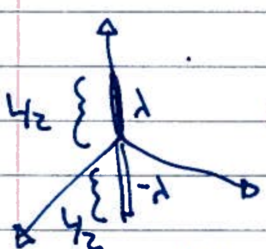
To find A we use Gauss law. Setting $\vec{r}'=0$:

$$\nabla^2 G(r) = \nabla\cdot\underbrace{\nabla G(r)}_{-\frac{A}{r^2}\hat{r}} \Rightarrow \underbrace{\oint da\,\hat{n}\cdot\nabla G}_{4\pi r^2\left(-\frac{A}{r^2}\right)} = -4\pi \Rightarrow A=1$$

$$\boxed{G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}}$$

Note: The relation $\nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\,\delta(\vec{r}-\vec{r}')$ is very usefull.

EXAMPLE: charged wire



$$\phi(r,\theta,z) = \int_0^{L/2} dz' \frac{\lambda}{\sqrt{r^2+(z-z')^2}} + \int_{-L/2}^{0} dz' \frac{(-\lambda)}{\sqrt{r^2+(z-z')^2}}$$

$$|\vec{r}-\vec{r}'| = r^2+(z-z')^2$$

$$= \lambda \ln\left[\frac{\cancel{\ldots}}{\left(-L+2z+\sqrt{4r^2+(-L+2z)^2}\right)} \ldots \frac{1}{L+2z+\sqrt{4r^2+(L+2z)^2}}\right.$$

$$\simeq \frac{\lambda z L^3}{4(r^2+z^2)^{3/2}} + \frac{\lambda(2z^2-3r^2)L^4}{64(r^2+z^2)^{7/2}} + \Theta\left(\left(\frac{L}{\sqrt{r^2+z^2}}\right)^6\right)$$

- Multipole expansion

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{(r-r')^i (r-r')^i}} \xrightarrow[r'\to 0]{} \frac{1}{r}$$

$$\frac{\partial}{\partial r'^i} \cdot \frac{1}{|r-r'|} = -\frac{1}{2}\,\frac{-2(r-r')^i}{\sqrt{\ }^3} \xrightarrow[r'\to 0]{} \frac{r^i}{r^3}$$

$$\frac{\partial^2}{\partial r'^i \partial r'^j} \cdot \frac{1}{|r-r'|} = -\frac{\delta^{ij}}{|\ |^3} + \frac{3}{2}\,\frac{2(r-r')^i (r-r')^j}{|\ |^5} \xrightarrow[r'\to 0]{} \frac{3r^i r^j - r^2\delta^{ij}}{r^5}$$

$$\phi(\vec{r}) = \int d\vec{r}\,'\, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \simeq \int d\vec{r}\,'\,\rho(\vec{r}\,')\left[\frac{1}{r} + r'^i \frac{r^i}{r^3} + \frac{r'^i r'^j}{2}\left(\frac{3r^i r^j - r^2\delta^{ij}}{r^5}\right) + \cdots\right]$$

$$= \frac{1}{r}\underbrace{\int d\vec{r}\,'\,\rho(\vec{r}\,')}_{\substack{q \\ \text{(Total charge)}}} + \frac{\hat{r}}{r^2}\cdot \underbrace{\int d\vec{r}\,'\,\rho(\vec{r}\,')\,\vec{r}\,'}_{\substack{\vec{p} \\ \text{(electric dipole moment)}}} + \frac{\hat{r}^i \hat{r}^j}{2r^3}\underbrace{\int d\vec{r}\,'\,\rho(\vec{r}\,')\left(3r'^i r'^j - r'^2\delta^{ij}\right)}_{\substack{Q^{ij} \\ \text{(electric quadrupole moment)}}} + \cdots$$

i) if $q=0$, $\vec{p}$ is independent of origin of coordinates and so on

ii) $Q^{ij} = Q^{ji}$, $Q^{ii} = 0$ ⇨ 5 independent components, 3 describe position (rotation), 2 the shape

iii)

$q \neq 0$ $\quad$ $q=0, \vec{p} \neq 0$ $\quad$ $q=0, \vec{p}=0, \overleftrightarrow{Q} \neq 0$

another way of setting up the multipole expansion

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{1}{r}\left(\frac{r'}{r}\right)^l P_l(\cos\gamma) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{1}{r}\left(\frac{r'}{r}\right)^l Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

$r' < r$

angle between $\vec{r}$ and $\vec{r}\,'$

addition theorem

$\hat{r}'$ $\hat{r}$

$$\phi(\vec{r}) = \int d\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \underbrace{\int d\vec{r}\,' \rho(\vec{r}\,')\, r'^l\, Y^*_{lm}(\theta',\varphi')}_{l^{\text{th}}\text{ multipole} \equiv q_{lm}} \frac{4\pi}{2l+1}\frac{1}{r^{l+1}} Y_{lm}(\theta,\varphi)$$

$$q_{00} = \int d\vec{r}\,' \rho(\vec{r}\,') \underbrace{Y_{00}(\theta',\varphi')}_{1/\sqrt{4\pi}} = \frac{1}{\sqrt{4\pi}}\, q \qquad \} \text{ charge}$$

$$q_{10} = \int d\vec{r}\,' \rho(\vec{r}\,') \sqrt{\frac{3}{4\pi}}\, z' = \sqrt{\frac{3}{4\pi}}\, p_z$$

$$q_{1\pm1} = \int d\vec{r}\,' \rho(\vec{r}\,')\left(-\sqrt{\frac{3}{8\pi}}\right)(x' \mp i y') = -\sqrt{\frac{3}{8\pi}}\,(p_x \mp i p_y)$$

} dipole

$\vdots$

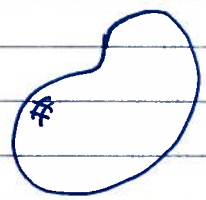
Y_{lm} form a basis for functions on the sphere.
Under rotations, Y_{lm} w/ different l's don't mix.

$\hat{r}$

most general problem; it includes problems 1. and 2.

PROBLEM 2. given charge distribution and $\underbrace{\text{non-trivial}}_{\neq\, \phi(\infty)=0}$ boundary conditions, find $\phi(\vec{r})$

TWO TYPES OF B.C.

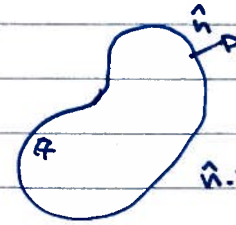


$\phi(S)$ given

($\phi(S) = \text{const} \Leftrightarrow$ conductor)

S

DIRICHLET



$\hat{n}$ $\quad \frac{\partial \phi}{\partial n} = \hat{n}\cdot\nabla\phi(S)$ given

$\hat{n}\cdot\nabla\phi = -\hat{n}\cdot\vec{E}$

$= -4\pi\sigma$

(if $\vec{E}=0$ in the interior)

NEUMANN

MATHEMATICAL INTERLUDE: GREEN'S IDENTITIES

$$\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi \Rightarrow \oint_{\partial V} da\, \hat{n}\cdot\phi\nabla\psi = \int_V d\vec{r}\,(\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi)$$

(1st Green identity)

(1st Green ident.) − (1st Green ident. w/ $\phi \leftrightarrow \psi$)

$$\Rightarrow \int_V d\vec{r}\,(\phi\nabla^2\psi - \psi\nabla^2\phi) = \oint_{\partial V} da\, \underbrace{\hat{n}\cdot(\phi\nabla\psi - \psi\nabla\phi)}_{\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}}$$

(2nd Green identity)

Some charges (in conductors) or polarizations (dielectrics) are not specified in practical situations and move depending on the value of $\vec{E}$. It's then better to specify b.c. and the position of "free" charges instead.

UNIQUENESS OF SOLUTION TO THE POISSON EQ. W/ BOUNDARY VALUES (DIRICHLET $\underline{\underline{\text{OR}}}$ NEUMANN, NOT BOTH) SPECIFIED ON A CLOSED SURFACE.

Take two sols. ϕ_1 and ϕ_2 and set $\phi = \psi = \phi_1 - \phi_2$ on the 1st Green ident.

$$\int d^3r \left[\underbrace{(\phi_1-\phi_2)\,\nabla^2(\phi_1-\phi_2)}_{-4\pi\rho + 4\pi\rho = 0} + |\nabla(\phi_1-\phi_2)|^2 \right] = \oint_{\partial V} da\, \underbrace{\hat{n}\cdot\left((\phi_1-\phi_2)\nabla(\phi_1-\phi_2)\right)}_{\underbrace{(\phi_1-\phi_2)}_{\substack{=0 \\ \text{if Dirichlet}}}\ \underbrace{\frac{\partial}{\partial n}(\phi_1-\phi_2)}_{\substack{=0 \\ \text{if Neuman}}}}$$

$$\Downarrow$$

$$\phi_1 = \phi_2 + \text{const.} \quad (\text{const} = 0 \text{ if Dirichlet})$$

EXAMPLES:

~~AN INTERESTING CASE~~

GREEN'S FUNCTIONS FOR BOUNDARY VALUE PROBLEMS (the analogue of Coulomb's law)

Set $\phi = \phi$ and $\psi = \frac{1}{|\vec{r}-\vec{r}\,'|}$ (for some $\vec{r}$) on the 2nd Green ident.:

$$\int_V d^3r' \left[\phi(\vec{r}\,')\, \underbrace{\nabla'^2 \frac{1}{|\vec{r}-\vec{r}\,'|}}_{-4\pi\delta(\vec{r}-\vec{r}\,')} - \frac{1}{|\vec{r}-\vec{r}\,'|} \underbrace{\nabla'^2 \phi(\vec{r}\,')}_{-4\pi\rho(\vec{r}\,')} \right] = \oint_{\partial V} da' \left[\phi \frac{\partial}{\partial n'} \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\vec{r}\,'|} \frac{\partial}{\partial n'} \phi(\vec{r}\,') \right]$$

$$\Downarrow$$

$$\boxed{\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \oint_{\partial V} da' \left[-\underbrace{\frac{\phi(\vec{r}\,')}{4\pi}}_{\substack{\text{dipole} \\ \text{surface density}}} \underbrace{\frac{\partial}{\partial n'} \frac{1}{|\vec{r}-\vec{r}\,'|}}_{\substack{\text{dipole} \\ \text{potential}}} + \frac{1}{|\vec{r}-\vec{r}\,'|} \underbrace{\frac{1}{4\pi} \frac{\partial \phi(\vec{r}\,')}{\partial n'}}_{\substack{\text{surface} \\ \text{charge} \\ (\text{if } \phi \text{ outside})}} \right]}$$

free charges

The eq. above is not a formal solution to the Poisson eq. because its use demand knowledge of both $\phi(S)$ and $\frac{\partial}{\partial n}\phi(S)$, an overspecification of the problem.

But we can modify this eq. by using a different Green's function instead of Coulomb law in order to take into account the induced charges and dipoles.

DIRICHLET PROBLEM : use $G_D(\vec{r},\vec{r}\,') = 0$ if $\vec{r}\,'$ in S

$\nabla^2_{r'} G_D(\vec{r},\vec{r}\,') = -4\pi\,\delta(\vec{r}-\vec{r}\,')$

$$\left(G_D(\vec{r},\vec{r}\,') = \underbrace{\frac{1}{|\vec{r}-\vec{r}\,'|}}_{\text{"free" charges}} + \underbrace{F(\vec{r},\vec{r}\,')}_{\text{induced charges}} \qquad \nabla^2_{r'} F = 0 \right)$$

$$\Downarrow$$

$$\phi(\vec{r}) = \int_V d\vec{r}\,'\, G_D(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,') - \oint_{\partial V} da\, \phi(\vec{r}\,')\, \frac{\partial}{\partial n'} G_D(\vec{r},\vec{r}\,')$$

given

zero here violates Gauss law

NEUMANN PROBLEM : use $\frac{\partial}{\partial n'} G_N(\vec{r},\vec{r}\,') = -\frac{4\pi}{A}$, $\vec{r}\,'$ in S (A is the area of S)

$\nabla^2_{r'} G_N(\vec{r},\vec{r}\,') = -4\pi\,\delta(\vec{r}-\vec{r}\,')$

$$\phi(\vec{r}) = \int_V d\vec{r}\,'\, G_N(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,') + \oint_{\partial V} da'\, G_N(\vec{r},\vec{r}\,')\,\frac{1}{4\pi}\frac{\partial \phi}{\partial n'} + \underbrace{\langle \phi \rangle_{\partial V}}_{\to 0}$$

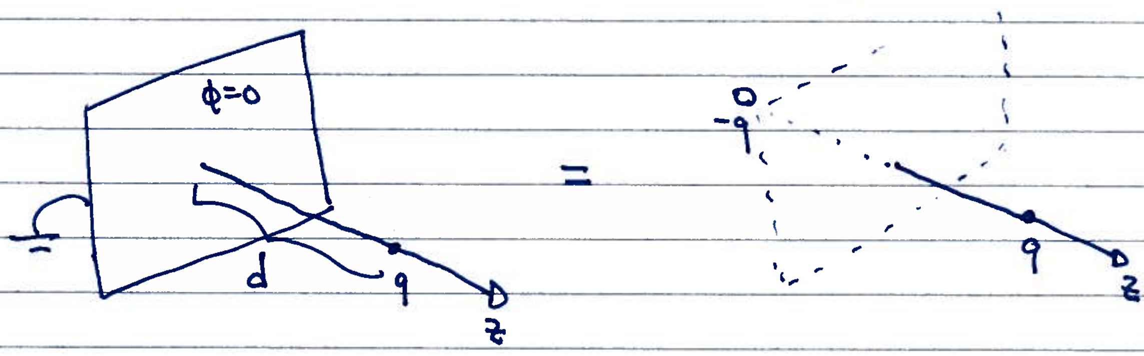
$\to 0$ if surface goes to infinity

G_N and G_D do <u>not</u> depend on either $\rho(\vec{r})$ and $\phi(S)$, $\frac{\partial \phi}{\partial n}(S)$, only on the geometry of the surface

A more general method to find G_D, G_N will be considered later. Let us look at some examples where a dirty trick suffices:

METHOD OF IMAGES

EXAMPLE: point charge + grounded conducting plane



$$\phi(x,y,z) = \frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}$$

EXAMPLE: point charge + grounded conducting sphere

$R\hat{r}$ = sphere

$$\phi(\vec{R}) = \frac{q}{|\vec{R}-d\hat{z}|} + \frac{q'}{|\vec{R}-d'\hat{z}|} = \frac{q}{R\underbrace{|\hat{r}-\frac{d}{R}\hat{z}|}_{1+\left(\frac{d}{R}\right)^2-2\frac{d}{R}\hat{r}\cdot\hat{z}}} + \frac{q'}{d'\underbrace{|\frac{\vec{R}}{d'}-\hat{z}|}_{1+\left(\frac{R}{d'}\right)^2-\frac{2R}{d'}\hat{r}\cdot\hat{z}}}$$

$$\Downarrow$$

$$\underbrace{\text{if } \frac{q}{R} = -\frac{q'}{d'} \text{ and } \frac{d}{R} = \frac{R}{d'}}_{d' = \frac{R^2}{d},\quad q' = -\frac{qd'}{R} = -\frac{q}{R}\frac{R^2}{d} = -\frac{qR}{d}} , \quad \phi(R\hat{r}) = 0$$

Induced charge: $E = 4\pi\sigma$ ($\phi = 0$ inside) $\Rightarrow \sigma = -\frac{1}{4\pi}\frac{\partial\phi}{\partial n} = -\frac{1}{4\pi}\hat{r}\cdot\nabla\phi$

$x = r\cos\theta\cos\varphi$

$y = r\cos\theta\sin\varphi$

$z = r\sin\theta$

$$\rho^2 = x^2+y^2+(z-d)^2 = \overbrace{r^2_{\cos\theta} + (r\sin\theta-d)^2}^{r^2+d^2-2dr\sin\theta} \xrightarrow{r\to R} R^2 + d^2 - 2dR\sin\theta = d^2\left[(R/d)^2 + 1 - 2R/d\sin\theta\right]$$

$$\rho'^2 = x^2+y^2+(z-d')^2 = \underbrace{r^2_{\cos\theta} + \left(r\sin\theta - \frac{R^2}{d}\right)^2}_{r^2+(R^2/d)^2-2r\frac{R^2}{d}\sin\theta} \xrightarrow[r\to R]{} R^2 + \left(\frac{R^2}{d}\right)^2 - 2\frac{R^3}{d}\sin\theta = R^2\left[1+\frac{R^2}{d^2}-2\frac{R}{d}\sin\theta\right]$$

$$= \frac{R^2}{d^2}\rho^2$$

$$\phi = q\left[\frac{1}{\rho} - \frac{R}{d}\frac{1}{\rho'}\right]$$

$$\sigma = -\frac{1}{4\pi}\frac{\partial}{\partial r}\phi = -\frac{q}{4\pi}\left[-\frac{1}{\rho^2}\frac{d\rho}{dr} + \underbrace{\frac{R}{d\rho'^2}}_{\frac{R}{d}\frac{d^2}{R^2\rho^2} = \frac{d}{R\rho^2}}\frac{d\rho'}{dr}\right]\Bigg|_{r=R}$$

$$\cancel{= -\frac{1}{4\pi}\frac{q}{\rho^2}\left[-\underbrace{\frac{d\rho}{dr}}_{2r-2d\sin\theta} + \frac{d}{R}\underbrace{\frac{d\rho'}{dr}}_{2r-2\frac{R^2}{d}\sin\theta}\right]\Bigg|_{r=R}}$$

$$\cancel{= -\frac{1}{4\pi}\frac{q}{R^2+d^2-2dR\sin\theta}\left[-2R+2d\sin\theta + 2d - 2R\sin\theta\right] = -\frac{1}{4\pi}2(d-R)(1+\sin\theta)}$$

$$\frac{d\rho}{dr} = \frac{d}{dr}\sqrt{r^2+d^2-2dr\sin\theta} = \frac{1}{2\rho}(2r-2d\sin\theta)$$

$$\frac{d\rho'}{dr} = \frac{1}{2\rho'}\left(2r-2\frac{R^2}{d}\sin\theta\right)$$

$$\sigma = -\frac{1}{4\pi}\cdot q\left[-\frac{1}{\rho^2}\frac{1}{2\rho}(2r-2d\sin\theta) + \frac{R}{d\,\rho'^2}\frac{1}{2\rho'}\left(2r-2\frac{R^2}{d}\sin\theta\right)\right]$$

$$= -\frac{q}{4\pi}\left[-\frac{1}{\rho^3}(r-d\sin\theta) + \frac{R}{d\rho'^3}\left(r-\frac{R^2}{d}\sin\theta\right)\right]$$

$$\frac{R}{d}\frac{d^3}{R^3}\frac{1}{\rho^3} = \frac{d^2}{R^2\rho^3}$$

$$= -\frac{q}{4\pi}\frac{1}{\rho^3}\left[-R+\cancel{d\sin\theta}+\frac{d^2}{R}-\cancel{d\sin\theta}\right]$$

$$= -\frac{q}{4\pi}\frac{d^2-R^2}{R}\frac{1}{(R^2+d^2-2Rd\sin\theta)^{3/2}}$$ ← $\sigma < 0$ everywhere

σ peaks @ $\theta = \pi/2$

EXAMPLE: point charge + conducting sphere w/ charge Q

Start w/ previous problem. Disconnect the ground. Add charge $Q-q'$ to the sphere. The $Q-q'$ charge distributes uniformly

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-d\hat{z}|} + \frac{q'}{|\vec{r}-d'\hat{z}|} + \frac{Q-q'}{r}$$

EXAMPLE: point charge + conducting sphere @ fixed potential V

As before w/ $Q-q' \to VR$. In fact,

$$\phi(r=R) = 0 + \frac{VR}{R} = V.$$

We essentially have found G_D for the sphere using the method of images. It's useful to rewrite in another way:

$$G_D(\vec r,\vec r\,') = \underbrace{\frac{1}{|\vec r-\vec r\,'|}}_{\text{charge}} \underbrace{- \frac{R}{r'}\frac{1}{|\vec r - \frac{R^2}{r'^2}\vec r\,'|}}_{\text{image (effect of induced charges)}}$$

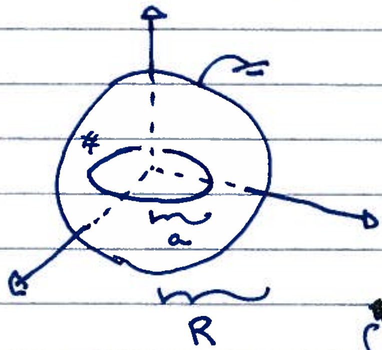
$$= F(\vec r,\vec r\,')$$

but
$$\frac{1}{|\vec r-\vec r\,'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

$r_<$ ($r_>$) is the smaller (larger) of r and r'

$$G_D(\vec r,\vec r\,') = \begin{cases} \displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \underbrace{\frac{R}{r'}\frac{1}{r}\left(\frac{R^2}{r'r}\right)^l}_{\frac{1}{R}\left(\frac{R^2}{rr'}\right)^{l+1}}\right]Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \\ r, r' > R \text{ (exterior)} \Rightarrow \frac{R^2}{r'} < r \\ \displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \underbrace{\frac{R}{r'}\frac{r'}{R^2}\left(\frac{rr'}{R^2}\right)^l}_{\frac{1}{R}\left(\frac{rr'}{R^2}\right)^{l}}\right]Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \\ r, r' < R \text{ (interior)} \Rightarrow \frac{R^2}{r'} > r \end{cases}$$

EXAMPLE: charged circular wire inside grounded conducting sphere



$$\rho(\vec{r}') = \frac{Q}{2\pi a^2}\,\delta(r'-a)\,\delta(\cos\theta')$$

spherical coord. δ-function

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi \int_0^{\infty} dr\, r^2 \underbrace{\frac{1}{r_0^2}\,\delta(r-r_0)\,\delta(\varphi-\varphi_0)\,\delta(\cos\theta-\cos\theta')}_{\delta(\vec{r}-\vec{r}')} = 1$$

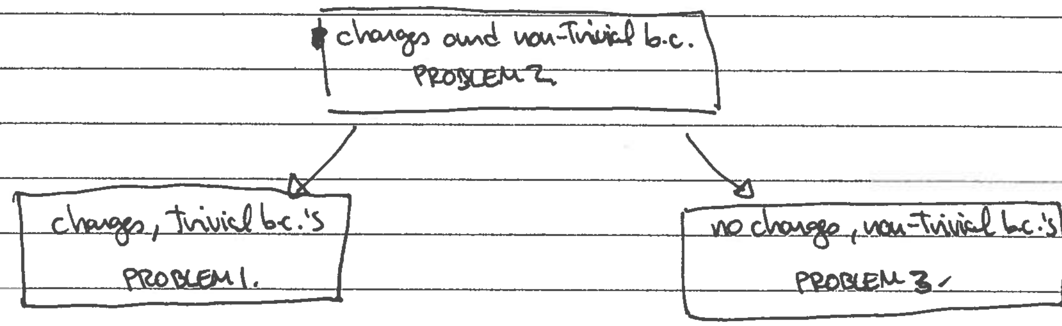
$$\phi(\vec{r}) = \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi \int_0^{R} dr\, r^2 \sum_{l,m} \frac{4\pi}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{R}\left(\frac{rr'}{R^2}\right)^l\right] Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

$$* \frac{Q}{2\pi a^2}\,\delta(r'-a)\,\delta(\cos\theta')$$

$$= Q \sum_{l} \frac{4\pi}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{R}\left(\frac{ar}{R^2}\right)^l\right] \underbrace{Y^*_{l0}(\pi/2, 0)}_{\sim\, P_l(0)}\ \underbrace{Y_{l0}(\theta,\varphi)}_{\sim\, P_l(\cos\theta)}$$

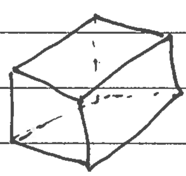
smallest, largest of a and r

PROBLEM 3. : non-Trivial b.c.'s but no charges



- ORTHOGONAL FUNCTIONS / SEPARATION OF VARIABLES

Cartesian coordinates (for "bricks")



$$\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\nabla^2\phi = \underbrace{\lambda\phi}_{\substack{\text{we want } \lambda=0 \text{ but}\\ \text{will consider general case}\\ \text{for future reference}}} \Longrightarrow \underbrace{\frac{X''}{X}}_{-k_x^2} + \underbrace{\frac{Y''}{Y}}_{-k_y^2} + \underbrace{\frac{Z''}{Z}}_{-k_z^2} = \lambda$$

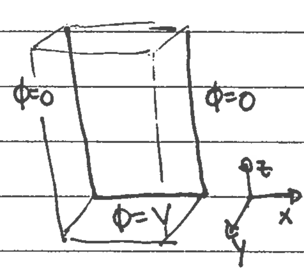
$$X(x) = A e^{ik_x x} + B e^{-ik_x x} = A' \sin k_x x + B' \cos k_x x$$

$$Y(y) = C e^{ik_y y} + D e^{-ik_y y} = \ldots$$

$$Z(z) = E e^{ik_z z} + F e^{-ik_z z} = \ldots$$

$$\vec{k}^2 = -\lambda$$ (for $\lambda=0$ this means one of $\vec{k}$ components is imaginary)

EXAMPLE:



$X(0) = X(L_1) = 0 \Rightarrow X(x) = \sin n\pi x/L_1$, $n=1,2,\ldots$

$Y(0) = Y(L_2) = 0 \Rightarrow Y(y) = \sin m\pi y/L_2$, $m=1,2,\ldots$

$$Z(z) = e^{-\overbrace{\left[\left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2\right]}^{\equiv \alpha_{nm}} z},\quad e^{+\left[\left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2\right] z}$$

$$\phi(x,y,z) = \sum_{n,m=1}^{\infty} \sin\frac{n\pi x}{L_1} \sin\frac{m\pi y}{L_2} \left[A_{nm} e^{-\alpha_{nm} z} + \underbrace{\cancel{B_{nm}}}_{0} e^{\alpha_{nm} z}\right]$$

$$\phi(x,y,0) = \sum_{n,m=1}^{\infty} \sin\frac{n\pi x}{L_1} \sin\frac{m\pi y}{L_2} \left[A_{nm} + \cancel{B_{nm}}^{0}\right] = V$$

$$A_{nm} = \int_0^{L_1} dx \int_0^{L_2} dy \; V \sin\frac{n\pi x}{L_1} \sin\frac{m\pi y}{L_2} \; \frac{2}{L_1} \frac{2}{L_2}$$

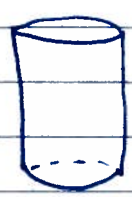
fix the normalization by imposing $\int_0^L dx \sin\frac{n\pi x}{L} \sin\frac{n'\pi x}{L} = \frac{L}{2}$

$$= \begin{cases} \frac{4V}{\cancel{L_1 L_2}} \frac{2\cancel{L_1 L_2}}{n\pi}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$\Downarrow$$

$$\boxed{\phi(x,y,z) = \sum_{\substack{n,m=1 \\ \text{odd}}}^{\infty} \frac{4V\sqrt{L_1 L_2}}{n\pi} \sin\frac{n\pi x}{L_1} \sin\frac{m\pi y}{L_2} \, e^{-\left[\left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2\right] z}}$$

Cylindrical coordinates (for cylinders, duh!)



$$\nabla^2 \phi = \lambda \phi$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2} = \lambda \phi$$

$$\phi = R(\rho)\, F(\varphi)\, Z(z)$$

$$\frac{1}{\rho R}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) + \frac{1}{\rho^2}\underbrace{\frac{1}{\phi}\frac{\partial^2 \phi}{\partial \varphi^2}}_{-\nu^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{+k^2} = \lambda$$

$Z'' - k^2 Z = 0 \Rightarrow Z(z) = e^{\pm kz}$ or $\cosh kz, \sinh kz$

$F'' + \nu^2 F = 0 \Rightarrow F(\varphi) = e^{\pm i\nu\varphi}, \; \nu = 0, 1, \dots \quad F(0) = F(2\pi)$

$R'' + \frac{1}{\rho} R' + \left(k^2 - \frac{\nu^2}{\rho^2} - \lambda\right) = 0 \Rightarrow J_\nu(k\rho), \; N_\nu(k\rho)$

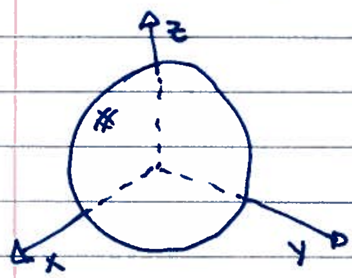
$\lambda = 0$

Bessel (regular @ $\rho=0$)

Neumann (diverges @ $\rho=0$)

$$\boxed{\phi(\rho,\varphi,z) = \sum_{n=1}^{\infty} \frac{\overset{2}{a}V\; J_0(x_{0n}\rho/a)\, \overset{\sinh}{\cancel{\cosh}}(x_{0n} z/a)}{x_{0n}\, \underset{\sinh}{\cancel{\cosh}}(x_{0n}L/a)\; J_1(x_{0n})}}$$

Spherical coordinates (good for spheres, who'd have thunk that!)



$$\nabla^2\phi = \lambda\phi$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = \lambda\phi$$

$$\phi(r,\theta,\varphi) = \frac{R(r)}{r}\,P(\theta)\,F(\varphi)$$

$$\underbrace{\frac{r}{R}\frac{1}{r}\frac{\partial^2 R}{\partial r^2}}_{+\frac{l(l+1)}{r^2}+\lambda} + \frac{1}{r^2\sin\theta}\frac{1}{P}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}P\right) + \frac{1}{r^2\sin^2\theta}\underbrace{\frac{1}{F}\frac{\partial^2 F}{\partial\varphi^2}}_{-m^2} = \lambda$$

$$\frac{1}{r^2}\underbrace{\left[\frac{1}{\sin\theta}\frac{1}{P}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right]}_{-l(l+1)}$$

$$\Downarrow$$

$$F''_{(\varphi)} + m^2F(\varphi) = 0 \Rightarrow F(\varphi) = e^{\pm im\varphi}, \quad m = 0,1,\ldots$$

$$\frac{1}{\sin\theta}(\sin\theta\,P')' + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0 \Rightarrow P(\theta) = P_l^m(\cos\theta), \quad l=0,1 \quad m=-l,\ldots,l$$

associated Legendre functions

$$R'' + \left[\lambda - \frac{l(l+1)}{r^2}\right]R(r) = 0 \underset{\lambda=0}{\Longrightarrow} R(r) = r^{l+1}, \frac{1}{r^l}$$

$$R \sim r^\alpha, \quad \alpha(\alpha-1)\cancel{r}^{\alpha-2} - l(l+1)\cancel{r}^{\alpha-2} = 0$$

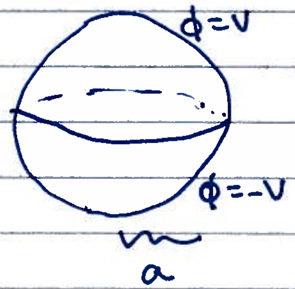
$$\alpha = l+1, -l$$

$$\sim P_\ell^m(\cos\theta)\, e^{+im\varphi}$$

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[a_{\ell m}\, r^{\ell+1} + \frac{b_{\ell m}}{r^{\ell}} \right] Y_{\ell m}(\theta,\varphi)$$

regular @ $r=0$ — diverges @ $r=0$

EXAMPLE: Two hemispheres @ different potentials



$b_{\ell m}=0$, $\phi(0,\theta,\varphi)$ is finite

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[a_{\ell m}\, r^{\ell+1} + \cancel{\frac{b_{\ell m}}{r^{\ell}}} \right] Y_{\ell m}(\theta,\varphi)$$

$$= \sum_{\ell=0}^{\infty} a'_{\ell 0}\, r^{\ell+1} P_\ell(\cos\theta)$$

axial symmetry

absorb the normalization of $Y_{\ell 0} \sim P_\ell$ in a'

b.c. @ $r=a$:

$$\phi(a,\theta,\varphi) = \begin{cases} V, & 0 \le \theta < \pi/2, \quad 0 \le \cos\theta \le 1 \\ -V, & \pi/2 \le \theta \le \pi, \quad -1 \le \cos\theta \le 0 \end{cases}$$

$$= \sum_{\ell=0}^{\infty} a'_{\ell 0}\, a^{\ell+1} P_\ell(\cos\theta)$$

$$a^{\ell+1} a'_{\ell 0} = \frac{2\ell+1}{2} \int_{-1}^{1} d\cos\theta \begin{cases} V, & 0 \le \cos\theta \le 1 \\ -V, & -1 \le \cos\theta < 0 \end{cases} P_\ell(\cos\theta)$$

$$\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}$$

$$\Downarrow$$

$$a'_{\ell 0} = \frac{(2\ell+1)V}{2a^{\ell+1}} \underbrace{\left[-\int_{-1}^{0} dx\, P_\ell(x) + \int_{0}^{1} dx\, P_\ell(x) \right]}_{\ell \to 0,\ 1,\ 0,\ -1/4,\ 0,\ 1/8,\ 0,\ -5/64}$$

$$\boxed{\phi(r,\theta,\varphi) = V\left[\frac{3r\cos\theta}{2a} - \frac{7}{8}\left(\frac{r}{a}\right)^3 \left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + \dots \right]}$$

Back to ~~problem~~ PROBLEM 3.: a "general" method of finding the Green's function

$$\nabla_r^2 G(\vec r,\vec r\,') = -4\pi\,\delta(\vec r-\vec r\,')$$

$$\Downarrow$$

$$G(\vec r,\vec r\,') = \sum_n \frac{\varphi_n(\vec r)\,\varphi_n^*(\vec r\,')}{\lambda_n}$$

EIGENVALUE PROBLEM

$\nabla_r^2 \varphi_n(\vec r) = \lambda_n \varphi_n(\vec r)$

eigenvalue problem w/ same b.c.'s as G

$\Downarrow$ ∇^2 is hermitian

$\int d\vec r\, \varphi_n^*(\vec r)\varphi_m(\vec r) = \delta_{nm}$

$\sum_n \varphi_n^*(\vec r)\varphi_n(\vec r\,') = \delta(\vec r-\vec r\,')$

λ_n are real

Indeed: $\nabla_{r'}^2 G(\vec r,\vec r\,') = \nabla_{r'}^2 \overset{(-4\pi)}{\sum_n} \dfrac{\varphi_n^*(\vec r)\,\varphi_n(\vec r\,')}{\lambda_n}$

$$= \overset{-4\pi}{\sum_n} \frac{\varphi_n^*(\vec r)\,\varphi_n(\vec r\,')}{\cancel{\lambda_n}}$$

$$= -4\pi\,\delta(\vec r-\vec r\,')$$

EXAMPLE: Trivial b.c. ($\phi(\infty)=0$)

$\nabla_r^2 \varphi_k(\vec r) = \lambda_k \varphi_k(\vec r) \Rightarrow \varphi_k(\vec r) = \dfrac{e^{i\vec k\cdot\vec r}}{(2\pi)^{3/2}}$, $\lambda_k = -k^2$ (normalization: $\int d\vec r\, \dfrac{e^{i\vec k\cdot\vec r}\, e^{-i\vec k'\cdot\vec r}}{2\pi} = \delta(\vec k-\vec k')$

$\int d\vec k\, \dfrac{e^{i\vec k(\vec r-\vec r\,')}}{(2\pi)^3} = \delta(\vec r-\vec r\,')$

$$G(\vec r,\vec r\,') = (-4\pi)\int \frac{d\vec k}{(2\pi)^3}\,\frac{e^{-i\vec k\cdot(\vec r-\vec r\,')}}{-k^2}$$

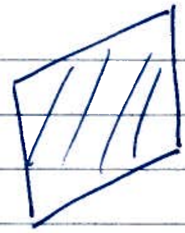
$$= 4\pi\int \frac{d^2k_\perp\, dk_\parallel}{(2\pi)^3}\,\frac{e^{-ik_\parallel|\vec r-\vec r\,'|}}{k_\perp^2+k_\parallel^2}$$

$$= 4\pi\,(-2\pi i)\int \frac{d^2k_\perp}{(2\pi)^3}\,\frac{e^{-k_\perp|\vec r-\vec r\,'|}}{-2ik_\perp}$$

$$= 4\pi\,\frac{1}{2}\int_0^\infty \frac{dk_\perp}{(2\pi)^2}\,\frac{2\pi \cancel{k_\perp}}{\cancel{k_\perp}}\, e^{-k_\perp|\vec r-\vec r\,'|}$$

$$= +\frac{\cancel{4\pi}}{\cancel{4\pi}}\,\frac{e^{-k_\perp|\vec r-\vec r\,'|}}{-|\vec r-\vec r\,'|}\Bigg|_0^\infty = +\frac{1}{|\vec r-\vec r\,'|}$$

EXAMPLE: $\phi(x, y, z=0) = 0$



$$\nabla_r^2 \varphi_k(\vec r) = \lambda_k \varphi_k(\vec r)$$

$$\Downarrow$$

$$\varphi_k(\vec r) = \sqrt{2}\,\frac{e^{i\vec k_\perp \cdot \vec r}}{(2\pi)^{3/2}} \sin(k_\parallel z)$$

normalization

$$\int \frac{d^2k_\perp\, dk_\parallel}{(2\pi)^2}\, e^{i\vec k_\perp \cdot (\vec r - \vec r\,')}\, 2 \left(e^{ik_\parallel z} - e^{-ik_\parallel z}\right)\left(-\frac{1}{4}\right)\left(e^{ik_\parallel z'} - e^{-ik_\parallel z'}\right)$$

$$= 2\,\delta^2(r_\perp - r_\perp')\left(-\frac{1}{4}\right)\int dk_\parallel \left[e^{ik_\parallel(z+z')} + e^{-ik_\parallel(z+z')} - e^{ik_\parallel(z-z')} - e^{-ik_\parallel(z-z')}\right]$$

$$= -\frac{2}{4}\,2\pi\,\delta^2(r_\perp - r_\perp')\, 2\left(\delta(z+z') - \delta(z-z')\right)$$

$$\underset{z, z' > 0}{=} \cancel{\tfrac{1}{4}}\, \delta^3(\vec r - \vec r\,')$$

$$G(\vec r, \vec r\,') = \int \frac{d^3k}{(2\pi)^3}\, 2\, e^{i\vec k_\perp \cdot (\vec r - \vec r\,')} \underbrace{\sin(k_\parallel z)\sin(k_\parallel z')}_{-\frac{1}{4}\left[e^{ik_\parallel(z+z')} + e^{-ik_\parallel(z+z')} - e^{ik_\parallel(z-z')} - e^{-ik_\parallel(z-z')}\right]}$$

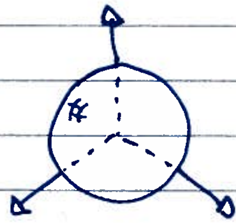
$$= 2\,\delta(\vec r_\perp - \vec r_\perp')\left(-\frac{1}{4}\right)\left(2\delta(z+z') - 2\delta(z-z')\right)$$

$$= +\,\delta(\vec r_\perp - \vec r_\perp')\left(\delta(z-z') - \delta(z+z')\right)$$

charge @ z'

image @ $-z'$

EXAMPLE: interior of a ~~the~~ sphere



$$\nabla^2 \varphi = \lambda \varphi$$

$$\varphi(r,\theta,\varphi) = \sum_{l,m} R_l(r) \; Y_{lm}(\theta,\varphi)$$

$$\left.\begin{array}{l} R_l'' + \left(\lambda - \frac{l(l+1)}{r^2}\right) R_l = 0 \\ \\ R_l(r\to 0) = \text{finite} \end{array}\right\} R(r) = \sqrt{r}\; J_{\frac{2l+1}{2}}(r\sqrt{\lambda})$$

$$R_l(a) = 0 \Rightarrow R(r) = \sqrt{r}\; J_{\frac{2l+1}{2}}(a\sqrt{\lambda}) = 0 \Rightarrow a\sqrt{\lambda_n} = x_{\frac{2l+1}{2}n}$$

$$\Downarrow$$

$$\varphi_{nlm}(r,\theta,\varphi) = \sqrt{r}\; J_{\frac{2l+1}{2}}\left(x_{\frac{2l+1}{2}n}\frac{r}{a}\right) Y_{lm}(\theta,\varphi)$$

$$\lambda_{nlm} = \left(\frac{x_{\frac{2l+1}{2}n}}{a}\right)^2$$

$$G(\vec{r},\vec{r}\,') = \sum_{n,l,m} \frac{a^2\sqrt{rr'}}{x^2_{l+\frac{1}{2}n}} \left[J_{\frac{2l+1}{2}}\left(x_{\frac{2l+1}{2}n}\frac{r}{a}\right) \; Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi) \; J_{\frac{2l+1}{2}}\left(x_{\frac{2l+1}{2}n}\frac{r'}{a}\right)\right]$$

$$a^2 \sum_{n=1}^{\infty} \frac{\sqrt{r r'}\, r'}{x^2_{\ell+\frac12 n}} J_{\ell+\frac12}(x_{\ell+\frac12 n} r/a)\, J_{\ell+\frac12}(x_{\ell+\frac12 n} r'/a) \stackrel{?}{=} \frac{r_<^\ell}{r_>^{\ell+1}} - \frac{1}{R}\left(\frac{rr'}{R^2}\right)^\ell$$

$$\cancel{\int_0^a dr'\, r' \left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{1}{R}\left(\frac{rr'}{R^2}\right)^\ell\right) J_{\ell+\frac12}(x_{\ell n} r'/a) \stackrel{?}{=}}$$

it must be true! One day I'll prove that, not today.

We can also find $G(\vec{r},\vec{r}')$ by solving the Poisson eq. in separable coordinates, like we solved the Laplace eq.:

EXAMPLE:

$$\nabla^2_{r'} G(\vec{r},\vec{r}') = -4\pi \underbrace{\delta(\vec{r}-\vec{r}')}_{\frac{1}{r'^2}\delta(r-r')\underbrace{\delta(\cos\theta-\cos\theta')\,\delta(\varphi-\varphi')}_{\sum_{\ell,m} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)}}$$

$$G(\vec{r},\vec{r}') = \sum_{\ell,m} \underbrace{g(r,\vec{r}')}\, Y_{\ell m}(\theta,\varphi)$$

$$\nabla^2_{r'} \sum_{\ell,m} g_{\ell m}(r,\vec{r}')\, Y_{\ell m}(\theta,\varphi) = -4\pi \frac{1}{r^2}\delta(r-r') \sum_{\ell,m} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)$$

$$\Downarrow$$

$$\nabla^2_{r'} g_{\ell m}(r,\vec{r}') = -\frac{4\pi}{r^2}\delta(r-r')\, Y^*_{\ell m}(\theta',\varphi')$$

$$\Downarrow$$

$$g_{\ell m}(r,\vec{r}') = g_{\ell m}(r,r')\, Y^*_{\ell m}(\theta',\varphi')$$

and $\displaystyle \frac{1}{r'}\frac{d^2}{dr'^2}\big(r\, g(r,r')\big) + \frac{1}{r^2}\big(-\ell(\ell+1)\big)\, g(r,r') = -\frac{4\pi}{r^2}\delta(r-r')$

$r<r'$: $\quad g(r,r') = A r'^\ell + \dfrac{B}{r'^{\ell+1}}$

$r>r'$: $\quad g(r,r') = C r'^\ell + \cancel{\dfrac{D}{r'^{\ell+1}}}$ well behaved @ $r'=0$

$$g(R, r') = 0 \Rightarrow C(R)\, r'^{\ell} = 0 \Rightarrow C(R) = 0$$

$$g(r, R) = 0 \Rightarrow A(r)\, R^{\ell} + \frac{B(r)}{R^{2\ell+1}} = 0 \Rightarrow B(r) = -A(r)\, R^{2\ell+1}$$

$$r < r' : \quad g(r, r') = A(r) \left[r'^{\ell} - \frac{R^{2\ell+1}}{r'^{\ell+1}} \right]$$

$$r > r' : \quad g(r, r') = C(r)\, r'^{\ell}$$

continuity: $g(r, r+\epsilon) = g(r, r-\epsilon) \Rightarrow A(r) \left[r^{\ell} - \frac{R^{2\ell+1}}{r^{2\ell+1}} \right] = C(r)\, r^{\ell}$

$$\Rightarrow C(r) = A(r) \left[1 - \frac{R^{2\ell+1}}{r^{2\ell+1}} \right]$$

$$= -\frac{4\pi}{2\ell+1} \left[\frac{r^{\ell}}{R^{2\ell+1}} - \frac{1}{r^{\ell+1}} \right]$$

(dis) continuity of derivative:

$$\underbrace{\int_{r-\epsilon}^{r+\epsilon} dr' \left[\frac{d^2}{dr'^2} (r' g(r, r')) - \cancel{\frac{\ell(\ell+1)}{r'^2} g(r, r')} \right]}_{\frac{d}{dr'} (r' g(r, r')) \Big|_{r' = r+\epsilon} - \frac{d}{dr'} (r' g(r, r')) \Big|_{r' = r-\epsilon}} = -\frac{4\pi}{r} \quad \to 0$$

$$A(r) \left[(\ell+1)\, r'^{\ell} + \frac{\ell R^{2\ell+1}}{r'^{\ell+1}} \right] - A(r) \left[1 - \frac{R^{2\ell+1}}{r^{2\ell+1}} \right] (\ell+1)\, r^{\ell} = -\frac{4\pi}{r}$$

$$A(r)\, \frac{R^{2\ell+1}}{r^{\ell+1}} \left[\ell + \ell + 1 \right] = -\frac{4\pi}{r} \Rightarrow A(r) = -\frac{4\pi}{r} \frac{1}{2\ell+1} \frac{r^{\ell+1}}{R^{2\ell+1}}$$

$$G(\vec{r}, \vec{r}') = \sum_{\ell, m} \frac{4\pi}{2\ell+1} \underbrace{\begin{cases} \left[-\frac{(r r')^{\ell}}{R^{2\ell+1}} + \frac{r^{\ell}}{r'^{\ell+1}} \right], & r < r' \\ \left[-\frac{(r r')^{\ell}}{R^{2\ell+1}} + \frac{r'^{\ell}}{r^{\ell+1}} \right], & r > r' \end{cases}}_{\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{(r r')^{\ell}}{R^{2\ell+1}}} Y^*_{\ell m}(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)$$

ELETROSTATICS IN MATTER

Suppose there's some polarization density $\vec{P}$ on a medium. It generates a potential

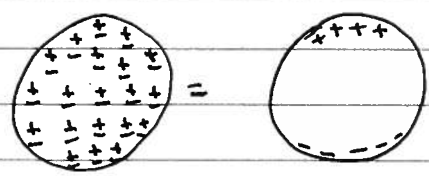
can I use the large r expansion here?
No. will justify later.

$$\phi(\vec{r}) = \int_V d\vec{r}' \left[\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \vec{P}(\vec{r}') \cdot \nabla_{r'} \frac{1}{|\vec{r}-\vec{r}'|} \right]$$

$$= \int_V d\vec{r}' \left[\frac{\underbrace{\rho(\vec{r}')}_{\text{"free" charge}} - \underbrace{\nabla_{r'} \cdot \vec{P}(\vec{r}')}_{\text{"bound" charge}}}{|\vec{r}-\vec{r}'|} \right] + \underbrace{\int_V d\vec{r}' \, \nabla_{r'} \cdot \left(\frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right)}_{\oint_{\partial V} da' \, \hat{n}' \cdot \vec{P}(\vec{r}') \frac{1}{|\vec{r}-\vec{r}'|}}$$

effective charge

surface density of bound charge



The relation between $\vec{E}$ and $\vec{P}$ changes a lot among different materials. One can even have $\vec{P}$ in the absence of $\vec{E}$ (electrets). If the relation is local, linear and isotropic

$$\vec{P} = \chi_e \vec{E}$$

~~polarizability~~
electric susceptibility

~~EXAMPLE: charge inserted in medium~~

Convenient to define $\vec{D}$

$$\nabla \cdot \vec{E} = 4\pi(\rho_{f} + \cancel{\rho_{bound}}) = 4\pi\rho_f - 4\pi\nabla\cdot P$$

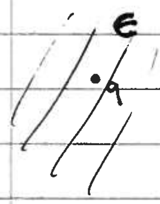
$$\Rightarrow \nabla \cdot \underbrace{(\vec{E} + 4\pi\vec{P})}_{\vec{D}} = 4\pi\rho_f$$

$$\vec{D} = \underbrace{\epsilon}_{\text{dielectric constant}} \vec{E} \Rightarrow \vec{E} + 4\pi\alpha\vec{E} = \epsilon\vec{E} \Rightarrow \epsilon = 1 + 4\pi\chi_e$$

$$\nabla \cdot \vec{D} = 4\pi\rho_f \qquad \nabla \times E = 0$$

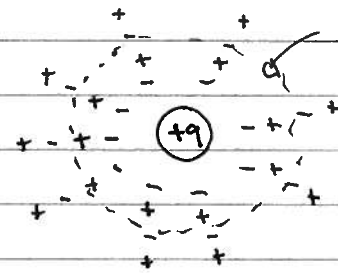
Gauss' law for $\vec{D}$ and ρ_f

EXAMPLE: charge in dieletric



$$\nabla\cdot(\epsilon\vec{E}) = 4\pi\rho \Rightarrow \nabla\cdot\vec{E} = \frac{4\pi\rho}{\epsilon} \Rightarrow \vec{E} = \frac{q}{\epsilon r^2}\hat{r}$$

$$\nabla\times\vec{E} = 0$$

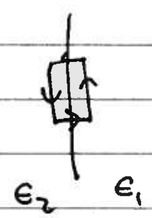


contains charge smaller than q.
(screening)

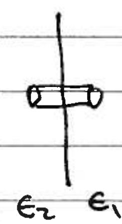
True (quantum) vacuum polarizes too (virtually). In QED there's screening and, ~~if we look at small enough scales~~ if we want the total charge inside the sphere to be fixed ~~...~~ as we shrink the sphere, the "bare" charges diverges at a finite radius! In non-abelian generalizations things are even stronger: there's anti-screening.

boundary conditions for $\vec{E}, \vec{D}$:

$\nabla\times\vec{E} = 0$



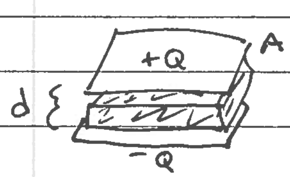
$$E^1_\parallel = E^2_\parallel$$
$$\frac{1}{\epsilon_1}D^1_\parallel = \frac{1}{\epsilon_2}D^2_\parallel$$

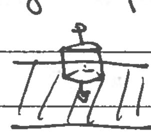


$$D^1_\perp = D^2_\perp$$
$$\epsilon_1 E^1_\perp = \epsilon_2 E^2_\perp$$

BOUNDARY ~~FREE~~ W/O FREE CHARGES

EXAMPLE: charged capacitor w/ dieletric



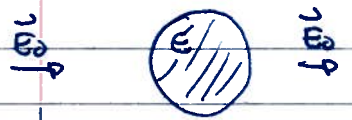


$4\pi\frac{Q}{A} = D$ (in between plates) $\Rightarrow E_{in} = \frac{4\pi Q}{\epsilon A}$

$0 = D$ (outside) $\Rightarrow E_{out} = 0$

$$P = \frac{D-E}{4\pi} = \frac{D(1-\epsilon)}{4\pi} = \begin{cases} \frac{Q}{A}(1-\epsilon) & \text{inside} \\ 0 & \text{outside} \end{cases}$$

EXAMPLE: dielectric sphere in an external field



inside: $\phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ (axial symmetry, no φ-dep., only $m=0$)

outside: $\phi_{out} = \sum_{l=0}^{\infty} \left[B_l r^l + \frac{C_l}{r^{l+1}} \right] P_l(\cos\theta)$

boundary conditions: $r \to \infty$, $\phi = -E_0 z = -E_0 \underbrace{r\cos\theta}_{\sim P_1(\cos\theta)} \Rightarrow B_1 = -E_0,\ B_l = 0$ for $l \neq 1$

$r = a$,

$$\underbrace{-\frac{1}{a}\frac{\partial \phi_{in}}{\partial \theta}}_{E_\parallel} = \underbrace{-\frac{1}{a}\frac{\partial}{\partial\theta}\phi_{out}}_{E_\parallel} \Rightarrow A_l a^l \frac{d}{d\theta}P_l(\cos\theta) = \left[B_l a^l + \frac{C_l}{a^{2l+1}}\right]\frac{dP}{d\theta}$$

$$\Rightarrow A_l = \frac{C_l}{a^{2l+1}}, \quad l \neq 1$$

$$A_1 = -E_0 + \frac{C_1}{a^3}, \quad l = 1$$

$$\underbrace{-\epsilon \frac{\partial}{\partial r}\phi_{in}}_{\epsilon E_\perp} = \underbrace{-\frac{\partial}{\partial r}\phi_{out}}_{E_\perp} \Rightarrow \epsilon\, l A_l a^{l-1} = l B_l r^{l-1} - (l+1)\frac{C_l}{a^{l+2}}$$

$$\Rightarrow \epsilon A_l = -\frac{l+1}{l}\frac{C_l}{a^{2l+1}}, \quad l \neq 1$$

$$\epsilon A_1 = -E_0 - \frac{2C_1}{a^3}, \quad l = 1$$

$\Downarrow$

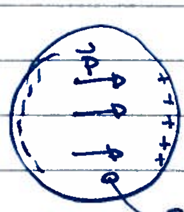
$A_l = C_l = 0, \quad l \neq 1$

$2A_1 + \epsilon A_1 = -3E_0$ or $A_1 = -\frac{3}{2+\epsilon}E_0$ and $C_1 = a^3 \underbrace{\left(\frac{-3+2+\epsilon}{2+\epsilon}\right)}_{\frac{\epsilon-1}{\epsilon+2}}$

$$\phi_{in} = -\frac{3}{2+\epsilon}E_0 \underbrace{r\cos\theta}_{z}$$

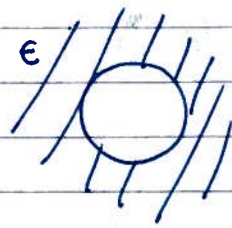
$$\phi_{out} = -E_0 r\cos\theta + \left(\frac{a^3}{r^2}\right)\frac{\epsilon-1}{\epsilon+2}E_0\cos\theta$$

$\cancel{\phi_{in} = \frac{3\epsilon}{1+2\epsilon}E_0}$ $\cancel{P = \frac{3}{4\pi}\frac{1-\epsilon}{1+2\epsilon}E_0}$ $\cancel{E_{in} = \epsilon E_0 + \frac{3}{3}\frac{1-\epsilon}{1+2\epsilon}E_0 = \frac{\epsilon+2\epsilon^2+1-\epsilon}{1+2\epsilon}E_0 = \frac{2+\epsilon}{1+2\epsilon}E_0}$



$E_{in} = \frac{3}{2+\epsilon}E_0 < E_0$, $P = \overbrace{\frac{\epsilon-1}{4\pi}}^{\chi_e} E_{in} = \frac{3}{4\pi}\frac{\epsilon-1}{\epsilon+2}E_0$ (constant)

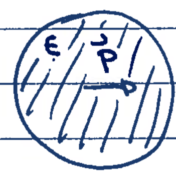
EXAMPLE: spherical cavity in a dielectric



as before with $\epsilon \to 1/\epsilon$ in the b.c. @ $r=a$

$$E_{in} = \frac{3\epsilon}{\epsilon+2} E_0 > E_0$$

EXAMPLE: dielectric sphere uniformly polarized



Take EXAMPLE 1, freeze the bound charges in place (fix $\vec{P}$) and subtract the external field E_0

$$E_0 = \frac{4\pi}{3}\frac{\epsilon+2}{\epsilon-1} P \Rightarrow$$

$$E_{in} = \frac{3}{2+\epsilon} E_0 - E_0 = \frac{3-2-\epsilon}{2+\epsilon} E_0 = \frac{1-\epsilon}{2+\epsilon}\frac{4\pi}{3}\frac{\epsilon+2}{\epsilon-1} P = -\frac{4\pi}{3} P$$

constant field inside and independent of a!

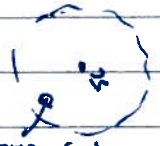
$$\phi_{out} = \frac{a^3}{r^2}\frac{\epsilon-1}{\epsilon+2}\frac{4\pi}{3}\frac{\epsilon+2}{\epsilon-1}\cos\theta\, P$$

$$= \frac{\cancel{a}}{r^2}\cos\theta \underbrace{\frac{4\pi a^3}{3}P}_{p}$$

= field of a single dipole @ the center w/ the total polarization

NOTE:

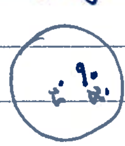
back to the $-\nabla\cdot P = \rho_{bound}$ result. What we should have done is to separate the integration in close and far from $\vec{r}$. On the outside ~~the~~ the dipole approximation is valid. The problem is to find the field generated by the dipoles close to $\vec{r}$. It turns out that any charge distribution within a sphere generates an average field on the sphere given by $\frac{4\pi}{3} P = \frac{p}{a^3}$. But, by the result above, the ~~total field~~ field ~~over the dipole fields~~ generated by a constant density of dipoles is the same.



dipole approx. valid

dipole approx. fails, but result is the same anyway

Proof: (Griffiths, problem 3.41). Take a charge q @ $\vec{r}'$ first, then use superposition



$$\underset{\text{average over sphere}}{\langle E \rangle_q} = \frac{1}{\frac{4\pi}{3}R^3}\int d\vec{r}\, \frac{q(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \underbrace{\frac{q}{\frac{4\pi}{3}R^3}\int d\vec{r}\, \frac{-\vec{r}'+\vec{r}}{|\vec{r}-\vec{r}'|^3}}_{\text{field produced by const density}} \overset{\text{Gauss law}}{=} \frac{q(\vec{r}-\vec{r}')}{R^3} = \frac{\vec{p}}{R^3}.$$

Clausius-Mossoti

$$\vec{p} = \alpha \vec{E}_m$$

$\vec{p}$: dipole moment of a molecule

$\vec{E}_m$: electric field @ the molecule position = applied + other polarized molecules

(dim. analysis: $[P] = [E] = \frac{[p]}{L^3} \Rightarrow [\alpha] \sim L^3$

$\alpha \sim a_0^3$)

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \epsilon\vec{E} \Rightarrow \vec{P} = \frac{\epsilon - 1}{4\pi}\vec{E}$$

$$\vec{E}_m = \underbrace{\vec{E}}_{\text{macroscopically averaged}} + \underbrace{\vec{E}_i}_{\text{fluctuation}} = \vec{E} + \underbrace{\vec{E}_{near}}_{\substack{\text{field due to molecules in}\\ \text{a little sphere around the molecule}\\ \simeq 0 \text{ (by symmetry)}}} - \underbrace{\vec{E}_p}_{\text{subtract field of dipoles}} = \vec{E} + \frac{4\pi}{3}\vec{P}$$

$$\vec{P} = \underbrace{N}_{\text{density of molecules}} \alpha \vec{E}_m = N\alpha\left(E + \frac{4\pi}{3}P\right) \Rightarrow P\left(1 - \frac{4\pi N\alpha}{3}\right) = N\alpha E$$

$$P = \underbrace{\frac{N\alpha}{1 - \frac{4\pi N\alpha}{3}}}_{\frac{\epsilon - 1}{4\pi}} E$$

$$4\pi N\alpha = \left(1 - \frac{4\pi N\alpha}{3}\right)(\epsilon - 1)$$

$$4\pi N\alpha \underbrace{\left(1 + \frac{\epsilon - 1}{3}\right)}_{\frac{\epsilon + 2}{3}} = \epsilon - 1$$

$$\boxed{\frac{4\pi}{3} N\alpha = \frac{\epsilon - 1}{\epsilon + 2}}$$ Clausius-Mossoti