Theoretical Dynamics September 24, 2010

Homework 3

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1 Goldstein 8.1

1.1 Part (a)

The Hamiltonian is given by

$$H(q_i, p_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \tag{1}$$

where all the $\dot{q}_i$'s on the RHS are to be expressed in terms of q_i, p_i and t. Now,

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt \tag{2}$$

From (1),

$$\begin{aligned} dH &= p_i d\dot{q}_i + \dot{q}_i dp_i - dL \\ &= p_i d\dot{q}_i + \dot{q}_i dp_i - \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt \right) \\ &= -\frac{\partial L}{\partial q_i} dq_i + \dot{q}_i dp_i + \left(p_i - \frac{\partial L}{\partial \dot{q}_i} \right) d\dot{q}_i - \frac{\partial L}{\partial t} dt \end{aligned} \tag{3}$$

Comparing (2) and (3) we get

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\dot{p}_i \qquad \text{(2nd equality from Hamilton's equation)} \tag{4}$$

$$\dot{q}_i = \frac{\partial H}{\partial q_i} \qquad \text{(also Hamilton's equation)} \tag{5}$$

$$p_i - \frac{\partial L}{\partial \dot{q}_i} = 0 \qquad \text{(H is not explicitly dependent on } \dot{q}_i\text{)} \tag{6}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{7}$$

From (4) and (6) we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, n \tag{8}$$

which are the Euler-Lagrange equations.

1.2 Part (b)

$$\begin{aligned} L'(p,\dot{p},t) &= -\dot{p}_i q_i - H(q,p,t) && (9)\\ &= p_i\dot{q}_i - H(q,p,t) - \frac{d}{dt}(p_i q_i) && (10)\\ &= L(q,\dot{q},t) - \frac{d}{dt}(p_i q_i) && (11)\\ &= L(q,\dot{q},t) - \dot{p}_i q_i - p_i \dot{q}_i && (12) \end{aligned}$$

So,

$$\begin{aligned} dL' &= \frac{\partial L'}{\partial p_i}dp_i + \frac{\partial L'}{\partial \dot{p}_i}d\dot{p}_i + \frac{\partial L'}{\partial t}dt && (13)\\ &= -\dot{q}_i dp_i - q_i d\dot{p}_i + \frac{\partial L}{\partial t}dt \qquad \text{(from (9))} && (14) \end{aligned}$$

Comparing (12) and (13) we get

$$\begin{aligned} \dot{q}_i &= -\frac{\partial L'}{\partial p_i} && (15)\\ q_i &= -\frac{\partial L'}{\partial \dot{p}_i} && (16) \end{aligned}$$

Thus the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{p}_i}\right) - \frac{\partial L'}{\partial p_i} = 0, \qquad i = 1,2,\ldots,n \tag{17}$$

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Hamilton's principle is

$$\delta \int L\,dt = 0 \tag{18}$$

or equivalently

$$\delta \int 2L\,dt = 0 \tag{19}$$

We can subtract the total time derivative of a function whose variation vanishes at the end points of the path, from the integrand, without invalidating the variational principle. This is because such a function will only contribute to boundary terms involving the variation of q_i and p_i at the end points of the path, which vanish by assumption. Such a function is $p_i q_i$. So, the 'modified' Hamilton's principle is

$$\delta \int \left(2L - \frac{d}{dt}(p_i q_i)\right) dt = 0 \tag{20}$$

Using the Legendre transformation, this becomes

$$\delta \int (2p_i\dot{q}_i - 2H - p_i\dot{q}_i - \dot{p}_i q_i)\ dt \quad = \quad 0 \tag{21}$$

$$\Longrightarrow \delta \int (2H + \dot{p}_i q_i - p_i\dot{q}_i)\ dt \quad = \quad 0 \tag{22}$$

now,

$$\dot{p}_i q_i - p_i\dot{q}_i \quad = \quad \left[\ q_1 \quad \ldots \quad q_n \quad | \quad p_1 \ldots \quad p_n\ \right]_{1\times 2n} \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}_{2n\times 2n} \begin{bmatrix} \dot{q}_1 \\ . \\ . \\ \dot{q}_n \\ -- \\ \dot{p}_1 \\ . \\ . \\ \dot{p}_n \end{bmatrix}_{2n\times 1} \tag{23}$$

$$= \quad \boldsymbol{\eta}^T \boldsymbol{J} \dot{\boldsymbol{\eta}} \tag{24}$$

So (22) becomes

$$\delta \int \left(2H + \boldsymbol{\eta}^T \boldsymbol{J} \dot{\boldsymbol{\eta}}\right)\ dt \quad = \quad 0 \tag{25}$$

which is the required form of Hamilton's principle.

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The constraints can be incorporated into the Lagrangian L by defining a "constrained Lagrangian" L_c, as

$$L_c(q,\dot{q},t) = L(q,\dot{q},t) - \sum_k \lambda_k \psi_k(q,p,t) \tag{26}$$

Applying Hamilton's principle, and using the Legendre transformation for L, we get

$$\delta \int \left(p_i\dot{q}_i - H(q,p,t) - \sum_k \lambda_k \psi_k(q,p,t) \right)\ dt \quad = \quad 0 \tag{27}$$

By analogy with the constrained Lagrangian, we can define a "constrained Hamiltonian" H_c as

$$H_c(q,p,t) = H(q,p,t) + \sum_k \lambda_k \psi_k(q,p,t) \tag{28}$$

Since both the terms are functions of q_i, p_i and t, this is a "good" Hamiltonian. Equation (27) can then be written as

$$\delta \int (p_i\dot{q}_i - H_c(q,p,t))\ dt \quad = \quad 0 \tag{29}$$

This bears a resemblance to the usual variational principle in Hamiltonian mechanics, for a Hamiltonian H_c. So the Hamilton equations are

$$\begin{aligned}\dot{q}_i &= \frac{\partial H_c}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H_c}{\partial q_i}\end{aligned}$$

which become

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + \sum_k \lambda_k \frac{\partial \psi_k}{\partial p_i} \tag{30}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} + \sum_k \lambda_k \frac{\partial \psi_k}{\partial q_i} \tag{31}$$

Time as a canonical variable

If time t is treated as a canonical variable, we define $q_{n+1} = t$. By Hamilton's equations

$$\dot{p}_{n+1} = -\frac{\partial H}{\partial q_{n+1}} \tag{32}$$

$$= -\frac{\partial H}{\partial t} \tag{33}$$

$$= -\frac{dH}{dt} \tag{34}$$

and

$$\dot{q}_{n+1} = \frac{\partial H}{\partial p_{n+1}} \tag{35}$$

$$= 1 \qquad (\text{since } q_{n+1} = t) \tag{36}$$

As the Hamiltonian contains terms of the form $p_i\dot{q}_i$ for each coordinate and its canonical momentum, in order to incorporate the constraint imposed by the inclusion of time as the $(n+1)^{th}$ canonical variable, we include a term of the form $p_{n+1}\dot{q}_{n+1} = p_{n+1}$ to the Hamiltonian to set up the constraint. Equivalently, the constraint can be obtained by integrating equation (34) above, and is given by

$$H(q_1, \ldots, q_n, q_{n+1}; p_1, \ldots, p_n) + p_{n+1} = 0 \tag{37}$$

Hamilton's principle,

$$\delta \int (p_i \dot{q}_i - H)dt = 0 \tag{38}$$

can be written as

$$\delta \int (p_i \dot{q}_i - H)t' d\theta = 0 \tag{39}$$

where $t' = dt/d\theta$ and θ is some parameter.

Using the constrained form of Hamilton's equations we get

$$\begin{aligned}
\dot{q}_i &= (1+\lambda)\frac{\partial H}{\partial p_i}, \qquad i = 1, 2, \ldots n && (40)\\
\dot{p}_i &= -(1+\lambda)\frac{\partial H}{\partial q_i}, \qquad i = 1, 2, \ldots n && (41)\\
\dot{q}_{n+1} &= \lambda && (42)\\
\dot{p}_{n+1} &= -(1+\lambda)\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} && (43)
\end{aligned}$$

By regarding $H' = (1+\lambda)H$ as an equivalent Hamiltonian, these equations are the required $(2n+2)$ equations of motion. Also, $\lambda = \dot{q}_{n+1} = dt/d\theta$.

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4.1 Part (a)

In the given configuration, both springs elongate or compress by the same magnitude. Suppose q denotes the position of the mass m from the left end. At $t = 0$, $q(0) = a/2$, but the unstretched lengths of both springs are given to be zero. Therefore, the elongation (compression) of spring k_1 is q and the compression (elongation) of spring k_2 is q. The potential energy is

$$V = \frac{1}{2}k_1 q^2 + \frac{1}{2}k_2 q^2 = \frac{1}{2}(k_1 + k_2)q^2 \tag{44}$$

The kinetic energy is

$$T = \frac{1}{2}m\dot{q}^2 \tag{45}$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}(k_1 + k_2)q^2 \tag{46}$$

The momentum canonically conjugate to the coordinate q is

$$p_q = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{47}$$

So the Hamiltonian is

$$H = p_q\dot{q} - L = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}(k_1 + k_2)q^2 \tag{48}$$

that is,

$$H(q, p_q, t) = \frac{p_q^2}{2m} + \frac{1}{2}(k_1 + k_2)q^2 \tag{49}$$

Clearly, the Hamiltonian equals the total energy E. The energy is conserved since,

$$\frac{dE}{dt} = m\dot{q}\ddot{q} + (k_1 + k_2)q\dot{q} = \dot{q}(-(k_1 + k_2)q) + (k_1 + k_2)q\dot{q} = 0 \tag{50}$$

where we have used the equation of motion[1]. In this case, the Hamiltonian is also conserved.

[1] $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \implies m\ddot{q} + (k_1 + k_2)q = 0$

4.2 Part (b)

Substituting $q = Q + b\sin(\omega t)$ and $\dot{q} = Q + b\omega\cos(\omega t)$ into the expression for the Lagrangian, we get

$$L(Q, \dot{Q}, t) = \frac{1}{2}m(\dot{Q} + b\omega\cos(\omega t))^2 - \frac{1}{2}(k_1 + k_2)(Q + b\sin(\omega t))^2 \tag{51}$$

and the momentum canonically conjugate to the coordinate Q is given by

$$p_Q = \frac{\partial L}{\partial \dot{Q}} = m(\dot{Q} + b\omega\cos(\omega t)) \tag{52}$$

So the Hamiltonian becomes

$$\begin{aligned}
H(Q, p_Q, t) &= p_Q\dot{Q} - L(Q, \dot{Q}, t) \qquad (53)\\
&= m(\dot{Q} + b\omega\cos(\omega t))\dot{Q} - \frac{1}{2}m(\dot{Q} + b\omega\cos(\omega t))^2 + \frac{1}{2}(k_1 + k_2)(Q + b\sin(\omega t))^2\\
&= \frac{m\dot{Q}^2}{2} - \frac{mb^2\omega^2}{2}\cos^2(\omega t) + \frac{1}{2}(k_1 + k_2)(Q + b\sin(\omega t))^2\\
&= \frac{p_Q^2}{2m} - p_Q b\omega\cos(\omega t) + \frac{1}{2}(k_1 + k_2)(Q + b\sin(\omega t))^2 \qquad (54)
\end{aligned}$$

The Hamiltonian is now explicitly dependent on time, and hence is not conserved, as is confirmed by the fact that $dH/dt \neq 0$. The energy is given by

$$E = T + V = \frac{1}{2}(\dot{Q} + b\omega\cos(\omega t))^2 + \frac{1}{2}(k_1 + k_2)(Q + b\omega\sin(\omega t))^2 \tag{55}$$

So,

$$\begin{aligned}
\frac{dE}{dt} &= m(\dot{Q} + b\omega\cos(\omega t))(\ddot{Q} - b\omega^2\sin(\omega t)) + (k_1 + k_2)(Q + b\sin(\omega t))(\dot{Q} + b\omega\cos(\omega t))\\
&= (\dot{Q} + b\omega\cos(\omega t))(m(\ddot{Q} - B\omega^2\sin(\omega t)) + (k_1 + k_2)(Q + b\sin(\omega t)))\\
&= (\dot{Q} + b\omega\cos(\omega t))(m\ddot{q} + (k_1 + k_2)q) \qquad (56)\\
&= 0 \qquad \text{(c.f. footnote on prev. page)} \qquad (57)
\end{aligned}$$

Therefore, the energy is conserved, as expected (the energy is still given by $T + V$, but the Hamiltonian is not $T + V$ anymore, as the relationship connecting the generalized coordinate to the cartesian coordinate is now explicitly dependent on time).

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5.1 Part (a)

The Lagrangian for the system is

$$L = \frac{1}{2}m(\boldsymbol{v} \cdot \boldsymbol{v}) + e\boldsymbol{A}(r) \cdot \boldsymbol{v} - eV(r) \tag{58}$$

The canonical momentum is

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + e\boldsymbol{A} \tag{59}$$

So the Hamiltonian is

$$\begin{aligned}
H &= \boldsymbol{p}\cdot\boldsymbol{v} - L && (60)\\
&= (m\boldsymbol{v}+e\boldsymbol{A})\cdot\boldsymbol{v} - \left(\frac{1}{2}m(\boldsymbol{v}\cdot\boldsymbol{v}) + e\boldsymbol{A}(r)\cdot\boldsymbol{v} - eV(r)\right)\\
&= \frac{m}{2}\boldsymbol{v}\cdot\boldsymbol{v} + eV(r)\\
&= \frac{(\boldsymbol{p}-e\boldsymbol{A})\cdot(\boldsymbol{p}-e\boldsymbol{A})}{2m} + eV(r) && (61)\\
&= \frac{1}{2m}(\boldsymbol{p}^2 - 2e\boldsymbol{p}\cdot\boldsymbol{A} + e^2\boldsymbol{A}^2) + eV(r) && (62)
\end{aligned}$$

Now,

$$\begin{aligned}
\boldsymbol{p}\cdot\boldsymbol{A} &= \boldsymbol{p}\cdot\frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}\\
&= \frac{1}{2}\boldsymbol{B}\cdot(\boldsymbol{r}\times\boldsymbol{p}) && (63)\\
&= \frac{1}{2}\boldsymbol{B}\cdot\boldsymbol{J} && (64)
\end{aligned}$$

where $\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{p}$ denotes the angular momentum. Also,

$$\begin{aligned}
\boldsymbol{A}^2 &= \frac{1}{4}(\boldsymbol{B}\times\boldsymbol{r})\cdot(\boldsymbol{B}\times\boldsymbol{r})\\
&= \frac{1}{4}B^2r^2 \qquad \text{(as } \boldsymbol{B} \text{ is perpendicular to } \boldsymbol{r}\text{)} && (65)
\end{aligned}$$

So the Hamiltonian of equation (58) becomes

$$H = \frac{\boldsymbol{p}^2}{2m} - \frac{e}{2m}\boldsymbol{B}\cdot\boldsymbol{J} + \frac{e^2}{8m}B^2r^2 + eV(r) \tag{66}$$

5.2 Part (b)

Let $\boldsymbol{v_{lab}} = (\dot{x},\dot{y})$ denote the velocity of the particle in the lab frame, and $\boldsymbol{v'} = (\dot{x}',\dot{y}')$ denote the velocity in the rotating frame. Without loss of generality, we may assume that motion is confined to the xy-plane. We first derive a relationship between the Hamiltonian in a rotating frame with that in a non-rotating frame (in this case, the lab frame). The coordinates are related by

$$\begin{aligned}
x &= x'\cos(\omega t) - y'\sin(\omega t) && (67)\\
y &= x'\sin(\omega t) + y'\cos(\omega t) && (68)
\end{aligned}$$

Here, it has been assumed that the rotation is counterclockwise, i.e. $\omega > 0$ for counterclockwise rotation. So the velocity components are related by

$$\begin{aligned}
\dot{x} &= \dot{x}'\cos(\omega t) - \dot{y}'\sin(\omega t) - \omega(x'\sin(\omega t) + y'\cos(\omega t)) && (69)\\
y &= \dot{x}'\sin(\omega t) + \dot{y}'\cos(\omega t) - \omega(x'\cos(\omega t) - y'\sin(\omega t)) && (70)
\end{aligned}$$

Therefore

$$\boldsymbol{v_{lab}}^2 = \dot{x}^2 + \dot{y}^2 \quad = \quad \dot{x}'^2 + \dot{y}'^2 + 2\omega(x\dot{y} - \dot{x}y) + \omega^2 r^2 \tag{71}$$

The Lagrangian in the lab frame is

$$\begin{aligned} L &= \frac{1}{2}m\boldsymbol{v_{lab}}^2 - eV(r) && (72)\\ &= \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2) + m\omega(x'\dot{y}' - \dot{x}'y') + \frac{1}{2}m\omega^2 r^2 - eV(r) && (73) \end{aligned}$$

The momenta canonically conjugate to x and y **in the rotating system** are

$$\begin{aligned} p_{x'} &= \frac{\partial L}{\partial \dot{x}'} = m(\dot{x}' - \omega y') && (74)\\ p_{y'} &= \frac{\partial L}{\partial \dot{y}'} = m(\dot{y}' + \omega x') && (75) \end{aligned}$$

So the Hamiltonian in the rotating frame is

$$\begin{aligned} H &= p_{x'}\dot{x}' + p_{y'}\dot{y}' - L && (76)\\ &= \frac{p_{x'}^2 + p_{y'}^2}{2m} + \omega(y'p_{x'} - x'p_{y'}) + eV(r) && (77)\\ &= \frac{p_{x'}^2 + p_{y'}^2}{2m} - J_z'\omega + eV(r) && (78) \end{aligned}$$

where J_z' denotes the angular momentum in the z-direction (direction of $\boldsymbol{\omega}$) as measured in the rotating frame. This means that for counterclockwise rotation along the z-axis,

$$H_{rotating\,frame} = H_{lab\,frame} - \omega J_z \tag{79}$$

This is the general relationship between Hamiltonians in the lab frame and rotating frame.

For this problem, from equation (62) above, we have

$$H_{lab\,frame} = \frac{\boldsymbol{p}^2}{2m} - \frac{eB}{2m}J + \frac{e^2}{8m}B^2 r^2 + eV(r) \tag{80}$$

as $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ and $\boldsymbol{J} = J_z\hat{\boldsymbol{z}} = J\hat{\boldsymbol{z}}$. So, the Hamiltonian in the rotating frame is

$$H_{rotating\,frame} = \frac{\boldsymbol{p}^2}{2m} - \left(\omega + \frac{eB}{2m}\right) J_z + \frac{e^2}{8m}B^2 r^2 + eV(r) \tag{81}$$

It is interesting to note that if $\omega = \omega_c = -\frac{eB}{2m}$, then the term linear in the magnetic field vanishes. In this problem, it is given that

$$\boldsymbol{\omega} = -\frac{e\boldsymbol{B}}{m} \tag{82}$$

which is twice the frequency ω_c. So, in this case, the Hamiltonian becomes

$$H_{rotating\,frame} = \frac{\boldsymbol{p}^2}{2m} + \frac{eB}{2m}J_z + \frac{e^2}{8m}B^2 r^2 + eV(r) \tag{83}$$

6 Problem 1

6.1 Part (a)

The subscript PB is suppressed for clarity.

$$\begin{aligned}
[L_i, L_j]_{PB} &= [\epsilon_{i\alpha\beta}x_\alpha p_\beta, \epsilon_{j\gamma\delta}x_\gamma p_\delta] \\
&= \epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}[x_\alpha p_\beta, x_\gamma p_\delta] \\
&= \epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}\left(x_\alpha[p_\beta, x_\gamma p_\delta] + [x_\alpha, x_\gamma p_\delta]p_\beta\right) \qquad (\text{as } [A, BC]_{PB} = B[A, C]_{PB} + [A, B]_{PB}C) \\
&= \epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}\left(x_\alpha[p_\beta, x_\gamma]p_\delta + x_\alpha x_\gamma[p_\beta, p_\delta] + [x_\alpha, x_\gamma]p_\delta p_\beta + x_\gamma[x_\alpha, p_\delta]p_\beta\right) \\
&= \epsilon_{i\alpha\beta}\epsilon_{j\gamma\delta}(-\delta_{\beta\gamma}x_\alpha p_\delta + \delta_{\alpha\delta}x_\gamma p_\beta) \\
&= \epsilon_{i\alpha\beta}\epsilon_{j\beta\delta}(-x_\alpha p_\delta) + \epsilon_{i\alpha\beta}\epsilon_{j\gamma\alpha}x_\gamma p_\beta \\
&= \epsilon_{i\alpha\beta}\epsilon_{j\delta\beta}x_\alpha p_\delta - \epsilon_{i\beta\alpha}\epsilon_{j\gamma\alpha}x_\alpha p_\beta \\
&= (\delta_{ij}\delta_{\alpha\delta} - \delta_{i\delta}\delta_{j\alpha})x_\alpha p_\delta - (\delta_{ij}\delta_{\beta\gamma} - \delta_{i\gamma}\delta_{j\beta})x_\gamma p_\beta \\
&= (\delta_{ij}x_\alpha p_\alpha - x_j p_i) - (\delta_{ij}x_\beta p_\beta - x_i p_j) \\
&= x_i p_j - x_j p_i
\end{aligned}$$

Now, $[L_1, L_2]_{PB} = x_1 p_2 - x_2 p_1 = L_3$, $[L_1, L_3]_{PB} = x_1 p_3 - x_3 p_1 = -L_2$, $[L_3, L_2]_{PB} = x_3 p_2 - x_2 p_3 = \epsilon_{321} - L_1$, etc. So, $x_i p_j - x_j p_i = \epsilon_{ijk}L_k$.

Hence, $[L_i, L_j]_{PB} = \epsilon_{ijk}L_k$.

6.2 Part (b)

For each $i = 1, 2, 3$,

$$\begin{aligned}
[L_i, L^2]_{PB} &= [L_i, L_j L_j] \qquad (\text{sum over } j) \\
&= L_j[L_i, L_j] + [L_i, L_j]L_j \\
&= L_j(\epsilon_{ijk}L_k) + (\epsilon_{ijk}L_k)L_j \\
&= 2\epsilon_{ijk}L_j L_k \\
&= 0 \qquad (\text{as } \epsilon_{ijk} \text{ is antisymmetric under } j \leftrightarrow k, \text{ while } L_j L_k \text{ is symmetric.})
\end{aligned}$$

So,

$$[\boldsymbol{L}, L^2]_{PB} = \hat{\boldsymbol{e}}_i[L_i, L^2]_{PB} = \mathbf{0} \tag{84}$$

6.3 Part (c)

For each $i = 1, 2, 3$,

$$[L_i, f(r)]_{PB} = \frac{\partial L_i}{\partial x_\alpha}\frac{\partial f(r)}{\partial p_\alpha} - \frac{\partial L_i}{\partial p_\alpha}\frac{\partial f(r)}{\partial x_\alpha} \tag{85}$$

Now, $r = \sqrt{x_i x_i}$ and $L_i = \epsilon_{ijk}x_j p_k$, so

$$\frac{\partial f}{\partial x_\alpha} = \frac{\partial r}{\partial x_\alpha}\frac{\partial f}{\partial r} = \frac{x_\alpha}{r}\frac{\partial f}{\partial r} \tag{86}$$

$$\frac{\partial f}{\partial p_\alpha} = 0 \tag{87}$$

So,

$$\begin{aligned}
[L_i, f(r)]_{PB} &= -\frac{\partial L_i}{\partial p_\alpha}\frac{\partial f(r)}{\partial x_\alpha} \\
&= -\frac{x_\alpha}{r}\frac{\partial(\epsilon_{ijk}x_j p_k)}{\partial p_\alpha}\frac{\partial f(r)}{\partial r} \\
&= -\frac{\epsilon_{ijk}x_\alpha x_j \delta_{\alpha,k}}{r}\frac{\partial f}{\partial r} \\
&= -\frac{\epsilon_{ijk}x_j x_k}{r}\frac{\partial f}{\partial r} \\
&= -\frac{(\boldsymbol{r}\times\boldsymbol{r})_i}{r}\frac{\partial f}{\partial r} \\
&= 0
\end{aligned} \tag{88}$$

7 Problem 2

7.1 Part (a)

$$[\xi_i, \xi_j] = \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial \xi_j}{\partial \eta_\beta} \tag{89}$$

So,

$$\begin{aligned}
\frac{d}{d\epsilon}[\xi_i, \xi_j] &= \frac{d}{d\epsilon}\left(\frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial \xi_j}{\partial \eta_\beta}\right) \\
&= \frac{d}{d\epsilon}\left(\frac{\partial \xi_i}{\partial \eta_\alpha}\right) J_{\alpha\beta} \frac{\partial \xi_j}{\partial \eta_\beta} + \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{d}{d\epsilon}\left(\frac{\partial \xi_j}{\partial \eta_\beta}\right) \\
&= \frac{\partial}{\partial \eta_\alpha}\left(\frac{d\xi_i}{d\epsilon}\right) J_{\alpha\beta} \frac{\partial \xi_j}{\partial \eta_\beta} + \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial}{\partial \eta_\beta}\left(\frac{d\xi_j}{d\epsilon}\right) \\
&= \frac{\partial}{\partial \eta_\alpha}\left([\xi_i, g]\right) J_{\alpha\beta} \frac{\partial \xi_j}{\partial \eta_\beta} + \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial}{\partial \eta_\beta}\left([\xi_j, g]\right) \\
&= \frac{\partial}{\partial \eta_\alpha}\left(\frac{\partial \xi_i}{\partial \eta_\gamma} J_{\gamma\delta} \frac{\partial g}{\partial \eta_\delta}\right) J_{\alpha\beta} \frac{\partial \xi_j}{\partial \eta_\beta} + \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial}{\partial \eta_\beta}\left(\frac{\partial \xi_j}{\partial \eta_\omega} J_{\omega\theta} \frac{\partial g}{\partial \eta_\theta}\right) \\
&= \left(\frac{\partial^2 \xi_i}{\partial \eta_\alpha \partial \eta_\gamma} J_{\gamma\delta} \frac{\partial g}{\partial \eta_\delta} + \frac{\partial \xi_i}{\partial \eta_\gamma} J_{\gamma\delta} \frac{\partial^2 g}{\partial \eta_\alpha \eta_\delta}\right) J_{\alpha\beta} \frac{\partial \xi_j}{\partial \eta_\beta} \\
&\quad + \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta}\left(\frac{\partial^2 \xi_j}{\partial \eta_\beta \partial \eta_\omega} J_{\omega\theta} \frac{\partial g}{\partial \eta_\theta} + \frac{\partial \xi_j}{\partial \eta_\omega} J_{\omega\theta} \frac{\partial^2 g}{\partial \eta_\beta \partial \eta_\theta}\right)
\end{aligned} \tag{90}$$

Now, for $\epsilon = 0$, $\boldsymbol{\xi} = \boldsymbol{\eta}$, so the second order terms

$$\left.\frac{\partial^2 \xi_i}{\partial \eta_\alpha \partial \eta_\gamma}\right|_{\epsilon=0}, \qquad \left.\frac{\partial^2 \xi_j}{\partial \eta_\beta \partial \eta_\omega}\right|_{\epsilon=0}$$

equal zero. So,

$$\left.\frac{d}{d\epsilon}[\xi_i,\xi_j]\right|_{\epsilon=0} = \frac{\partial \xi_i}{\partial \eta_\gamma} J_{\gamma\delta} \frac{\partial^2 g}{\partial \eta_\alpha \partial \eta_\delta} J_{\alpha\beta} \frac{\partial \xi_j}{\partial \eta_\beta} + \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial \xi_j}{\partial \eta_\omega} J_{\omega\theta} \frac{\partial^2 g}{\partial \eta_\beta \partial \eta_\theta} \tag{91}$$

$$= \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial^2 g}{\partial \eta_\gamma \partial \eta_\beta} J_{\gamma\delta} \frac{\partial \xi_j}{\partial \eta_\delta} + \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial^2 g}{\partial \eta_\gamma \partial \eta_\beta} J_{\delta\gamma} \frac{\partial \xi_j}{\partial \eta_\delta} \tag{92}$$

which is obtained by switching $\gamma \leftrightarrow \alpha$, $\delta \leftrightarrow \beta$ in the first term and $\theta \leftrightarrow \gamma$, $\omega \leftrightarrow \delta$ in the second term. As J is antisymmetric, $J_{\delta\gamma} = -J_{\gamma\delta}$. So,

$$\left.\frac{d}{d\epsilon}[\xi_i,\xi_j]\right|_{\epsilon=0} = \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial^2 g}{\partial \eta_\gamma \partial \eta_\beta} J_{\gamma\delta} \frac{\partial \xi_j}{\partial \eta_\delta} - \frac{\partial \xi_i}{\partial \eta_\alpha} J_{\alpha\beta} \frac{\partial^2 g}{\partial \eta_\gamma \partial \eta_\beta} J_{\gamma\delta} \frac{\partial \xi_j}{\partial \eta_\delta} \tag{93}$$

$$= 0 \tag{94}$$

So upto $O(\epsilon^2)$, we have $d[\xi_i,\xi_j]/d\epsilon = 0$ and so $[\xi_i,\xi_j]$ is constant up to $O(\epsilon^2)$. As $[\xi_i,\xi_j]|_{\epsilon=0} = [\eta_i,\eta_j] = J_{ij}$, we have $[\xi_i,\xi_j] = J_{ij}$ upto $O(\epsilon^2)$.

So, $\boldsymbol{\xi}$ is a canonical transformation upto $O(\epsilon^2)$.

7.2 Part (b)

First of all,

$$\boldsymbol{\eta}(\boldsymbol{\xi}, 0) = \boldsymbol{\xi} \tag{95}$$

because at $t = 0$, the canonical coordinates overlap in phase space. So, using the result of part (a), $\boldsymbol{\eta}(\boldsymbol{\xi}, t)$ can be treated as a one-parameter family of canonical transformations (parametrized by t), provided there exists some function g satisfying

$$\frac{d\xi_i}{dt} = [\xi_i, g]_{PB} \tag{96}$$

This condition is seen to be satisfied if g is taken as the Hamiltonian H, for in this case,

$$[\xi_i, H]_{PB} = \frac{\partial \xi_i}{\partial \xi_k} J_{kj} \frac{\partial H}{\partial \xi_j} \tag{97}$$

$$= \delta_{ik} J_{kj} \frac{\partial H}{\partial \xi_j} \tag{98}$$

$$= J_{ij} \frac{\partial H}{\partial \xi_j} \tag{99}$$

$$= \dot{\xi}_i \tag{100}$$

where the last equality is obtained via Hamilton's equations. So, taking $g = H$ satisfies the Poisson bracket relation with H acting as the generator of time translations.