Physics 601 Homework 1---Due Friday, September 10

Hint: For some of these problems it will be helpful to use Mathematica or some other symbolic manipulation program. If you make use of such a program please include the output with your homework solutions.

Goldstein 2.2, 2.4, 2.12

In addition:

1. Consider a particle that is constrained to move in 1 dimension and is confined to a 1-dimensional box of length L. The particle bounces elastically back and forth between the walls of the box. Thus at the wall there is an impulsive force which flips the velocity ($\dot{x} \to -\dot{x}$) conserving energy. This problem explores what happens if at time t=0 the particle has velocity v_0 and one of the walls is moved slowly (either inward or outward) so that size of the box is now a function of time L(t). Since the wall is moving it can add or remove energy from the system. The goal is to find an adiabatic invariant relating the energy to L.
   a. Show that when the particle hits the moving wall $\dot{x} \to -\dot{x} + 2\dot{L}$. (Hint, what does the process look like from wall's point of view.)
   b. Quantify what is meant by "slowly" in terms of the parameters of the problem.
   c. Show that $L^2 E$ is an adiabatic invariant.
   d. From your knowledge of the 1-dimenisonal particle in the box in quantum mechanics, explain why this result is expected.

2. Consider the Lagrangian for a simple 1-dimesnional harmonic oscillator: $L(x,\dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m\omega_0^2 x^2$where ω_0 is a parameter.

   a. Find the equation of motion from Lagrange's equations.

   Consider a change of variables to the generalized coordinate $q = \sinh^{-1}(x)$

   b. Using the result of a. , find the equation of motion for q.
   c. Find the Lagrangian for q (*i.e.* find $L(q,\dot{q})$).
   d. Find the Lagrange's equation of motion for $L(q,\dot{q})$.
   e. How do the results of b. and d. compare? Why is this expected?

3. Again consider a simple 1-dimensional harmonic oscillator with $L(x,\dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m\omega_0^2 x^2$. Consider a family of trajectories with $x(t)$ subject to the boundary conditions $x(0) = 0$, $x(T) = l$. Suppose further that the family of trajectories includes the solution of the exact equations of motion.

An example of such a family is the set of function $x(t,\omega) = \frac{l\sin(\omega t)}{\sin(\omega T)}$ where ω is a parameter.

a. Verify that this family satisfies the boundary conditions.
b. Show that for $\omega = \omega_0$ this trajectory correspond to the solution of the full equations of motion.
c. Calculate the action as a function of ω: *i.e.* find $S(\omega)$.
d. Show explicitly by calsulationg the action that $\frac{dS(\omega)}{d\omega} = 0$ at $\omega = \omega_0$. Explain why this is expected

4. Problem 2 considered a family of trajectories containing the exact solution of the equations of motion. Suppose one has a when a family of trajectories which does **not** contain the exact solution. The variational principal can still be of use in finding approximate solutions. Consider the harmonic oscillator of problem 2 with the same boundary conditions. Consider the family of trajectories given by $x(t,c) = l\frac{t}{T} + c\,l\left(\frac{t^3}{T^3} - \frac{t}{T}\right)$ where *c* is a parameter.

   a. Verify that this family satisfies the boundary conditions.
   b. Calculate the action as a function of *c*: *i.e.* find $S(c)$.
   c. Minimize $S(c)$ to find the "best" approximation to the full solution of the form considered.
   d. Plot the exact solution and the approximate solutions for three cases: $i) T\omega_0 = 1$ $\quad ii) T\omega_0 = 2$ $\quad iii) T\omega_0 = 3$. Qualitatively discuss the difference in these cases and why these differences make sense.