Solutions of Homework # 2

Problem 2.1

According to the equation(2.7) on the textbook, we have:

$$\frac{f_{quad}}{f_{lin}} = \frac{cv^2}{bv} = \left(1.6 \times 10^3 \frac{s}{m^2}\right) Dv$$

For the two types of forces are of the same strength, we have:

$$\left(1.6 \times 10^3 \frac{s}{m^2}\right) Dv = 1 \Rightarrow v = 8.9 \times 10^{-3} m/s$$

So when $v >> 10^{-2}$m/s, we could have f as pure f_{quad}, and it's good approximation to ignore the linear force in real case.

If we take $D = 0.7$m, we will have $v' = 8.9 \times 10^{-4}$m/s for the critical velocity for the equality of the two forces and it is good approximation to ignore the linear force in real case.

Problem 2.8

From the Newton's second law, we have $d[mv(t)] = F(v)dt$:

$$dt = m\frac{dv(t)}{F(v)} \Rightarrow t - t_0 = m\int_{v_o}^{v} \frac{dv}{-cv^{3/2}} = \frac{2m}{c}(v^{-1/2} - v_0^{-1/2})$$

$$\Rightarrow v = \frac{v_0}{\left[1 + \frac{c\sqrt{v_0}}{2m}(t - t_0)\right]^2}$$

So, when the mass is to rest, $v \to 0$, we have $\triangle t = t - t_0 \to \infty$.

Problem 2.13

From the conversation of energy, we have $d(\frac{1}{2}mv^2) = F(x)dx$:

$$dv^2 = \frac{2F(x)dx}{m} \Rightarrow v^2 - v_0^2 = -\int_{x_0}^{x} \frac{2kxdx}{m} = \frac{k(x_0^2 - x^2)}{m}$$

$$\Rightarrow v = -\sqrt{\frac{k(x_0^2 - x^2)}{m}}$$

where we have used $v_0 = 0$. The sign of v is in fact determined from the sign of x_0, have negative sign for v when $x_0 > 0$ and positive sign when $x_0 < 0$, and this discussion is available only for the first half cycle of the motion and the sign does not affect the expression of the x(t).

$$dt = \frac{dx}{v} \Rightarrow t = -\int_{x_0}^{x} \sqrt{\frac{m}{k}} \frac{dx}{(x_0^2 - x^2)^{1/2}} = \sqrt{\frac{m}{k}} \arccos\left(\frac{x}{x_0}\right)$$

$$\Rightarrow x = x_0 \cos\left(\sqrt{\frac{k}{m}} t\right)$$

Problem 2.21

In the polar coordinates, all ρ directions are same, that is our discussion has nothing to with ϕ. Let's take some ϕ_0 and this problem become 2D problem and its expansion to 3D problem is natural. Let's take $z_0 = 0$

Before the shell hit $z = 0$, we have the equations of motion:

$$z(t) = v_z t - \frac{1}{2} g t^2 \qquad \rho(t) = v_\rho t$$
$$\Rightarrow z(t) = \frac{v_z}{v_\rho}\rho(t) - \frac{1}{2} g \left(\frac{\rho(t)}{v_\rho}\right)^2$$

Now, let's analyze this problem. What does it mean to hit all objects inside a surface? It means that for a fixed ρ, the point on the surface should have the maximum z for all possible path passing this ρ, or for a fixed z, the point on the surface should have maximum ρ for all possible path passing this z. So let's denote $\frac{v_z}{v_\rho} = a$, and suppose ρ fixed, it turns out that $z = a\rho - \frac{g\rho^2}{2v_0^2}(a^2+1)$, and for this surface, we will have:

$$\frac{\partial z}{\partial a} = \rho - \frac{g\rho^2 a}{v_0^2} = 0 \Rightarrow a = \frac{v_0^2}{g\rho}$$
$$\Rightarrow z = a\rho - \frac{g\rho^2}{2v_0^2}(a^2+1) = \frac{v_0^2}{2g} - \frac{g\rho^2}{2v_0^2}$$

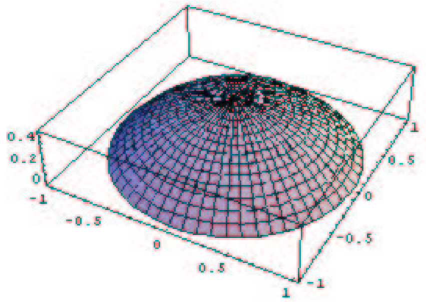

Figure 1

Problem 2.27

From the Newton's second law, we have the motion of puck on the upward journey:

$$m\frac{dv}{dt} = -cv^2 - mg\sin\theta$$

Let's denote $v_c^2 = mg\frac{\sin\theta}{c}$, so we have:

$$\frac{dv}{1+\frac{v^2}{v_c^2}} = -\frac{v_c c}{m}dt \Rightarrow v = v_c \tan\left[\arctan\left(\frac{v_0}{v_c}\right) - \frac{v_c c}{m}t\right]$$

For $v = 0$, we have $t = \frac{m}{v_c c}\arctan\left(\frac{v_0}{v_c}\right)$.

Problem 2.28

From the conversation of the energy $d(mv^2/2) = Fdx$, we have:

$$dx = \frac{mvdv}{F} = -\frac{mdv}{cv^{1/2}} \Rightarrow x - x_0 = \frac{2m}{c}(v_0^{1/2} - v^{1/2})$$

So, for v=0, we have $\triangle x = x - x_0 = \frac{2m}{c}v_0^{1/2}$

Problem 2.33

a. The figure of $\cosh z$ and $\sinh z$ when z are real is shown on the figure 2.

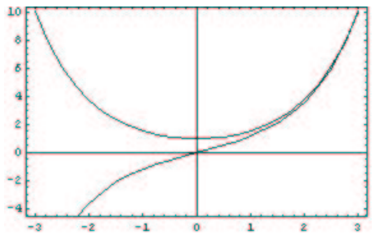

Figure 2

b.

$$\cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{z} = \frac{e^{z} + e^{-z}}{2} = \cosh z$$
$$\sin iz = \frac{e^{i(iz)} - e^{-i(iz)}}{z} = \frac{-e^{z} + e^{-z}}{2} = -\sinh z$$

c.

$$\frac{d\cosh z}{dz} = \frac{e^{z} - e^{-z}}{2} = \sinh z \qquad \frac{d\sinh z}{dz} = \frac{e^{z} + e^{-z}}{2} = \cosh z$$

d.

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^{z} + e^{-z}}{2}\right)^2 - \left(\frac{e^{z} - e^{-z}}{2}\right)^2 = 1$$

e. Suppose $x = \sinh z$, so we have:

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh z dz}{\cosh z} = z = \sinh^{-1} z$$

Problem 2.49

a.

$$\begin{aligned} z^2 &= \left(e^{i\theta}\right)^2 = e^{2i\theta} = \cos 2\theta + i \sin 2\theta \\ &= (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta \end{aligned}$$

So, we have $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$.

b.

$$\begin{aligned} z^3 &= \left(e^{i\theta}\right)^3 = e^{3i\theta} = \cos 3\theta + i\sin 3\theta \\ &= (\cos\theta + i\sin\theta)(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta) \\ &= (\cos^3\theta - 3\sin^2\theta\cos\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta) \end{aligned}$$

So, we have $\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$, $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$.

Problem 2.53

According to Newton's second law, we have:

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$
$$\Rightarrow m\dot{\boldsymbol{v}}_x = qv_yB, \quad m\dot{\boldsymbol{v}}_y = -qv_xB, \quad m\dot{\boldsymbol{v}}_z = qE$$

For the equation in x,y direction, it is easy to find the motion in xy plane is circle rotation. Introduce $\eta = v_x + iv_y$ and $\omega = qB/m$ we have:

$$m\dot{\eta} = -qB\eta \Rightarrow \eta = (v_{x0} + iv_{y0})e^{-i\omega t} \Rightarrow x + iy = (x_0 + iy_0)e^{-i\omega t}$$
$$\Rightarrow x = x_0\cos\omega t + y_0\sin\omega t, \quad y = y_0\cos\omega t - x_0\sin\omega t$$

Where we suppose z axis is chosen so that x,y has the above form. And the z direction is something like the motion in gravity field: $z = z_0 + v_{z0}t + \frac{qE}{m}t^2$. The combination of the motion in the three direction is a spiral motion.

Problem 2.55
a.

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$
$$\Rightarrow m\dot{\boldsymbol{v}}_x = qv_yB, \quad m\dot{\boldsymbol{v}}_y = qE - qv_xB, \quad m\dot{\boldsymbol{v}}_z = 0$$

From the equation of z direction, we have v_z is constant. Because $v_{z0} = 0$, we have $v_z = 0$. That is the motion is restricted in xy plane.
b. If we request the motion is undeflected, we will have the general form:

$$\frac{v_y(t)}{v_x(t)} = \frac{v_y(0)}{v_x(0)} = 0 \Rightarrow v_y(t) = 0$$
$$\Rightarrow m\dot{\boldsymbol{v}}_y = qE - qv_xB = 0, \quad m\dot{\boldsymbol{v}}_x = qv_yB = 0$$
$$\Rightarrow v = v_{x0} = \frac{E}{B}$$

c. Make the transformation: $v'_x = v_x - E/B$, so the equation of motion is:

$$m\dot{v}'_x = qv_yB, \quad m\dot{v}_y = -qv'_xB$$
$$\Rightarrow v'_x + iv_y = (v_{x0} - E/B)e^{-i(qB/m)t}$$
$$\Rightarrow v_x = (v_{x0} - E/B)\cos[(qB/m)t] + E/B, \quad v_y = -(v_{x0} - E/B)\sin[(qB/m)t]$$

The procedure of solving this equation is the same as the problem of 2.53.
d.

$$x = x_0 + \int_0^t v_x dt = E/Bt + \left(\frac{v_{x0} - E/B}{qB/m}\right)\sin(qBt/m)$$
$$y = y_0 + \int_0^t v_y dt = \left(\frac{v_{x0} - E/B}{qB/m}\right)(\cos(qBt/m) - 1)$$

Let's take qB/m=1, $E/B = 1$ and for v_{x0}=-10, -1, 0, 0.5, 2, 2.5, 3, 10, we have the graphs of trajectory in figure 3.

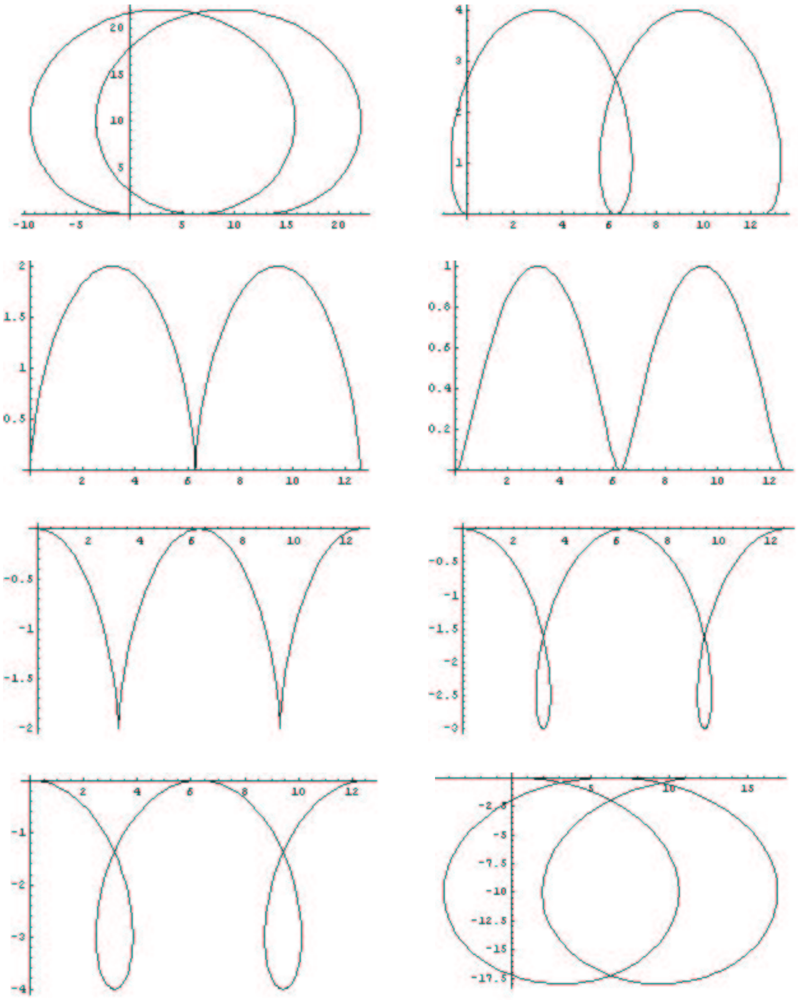

Figure 3