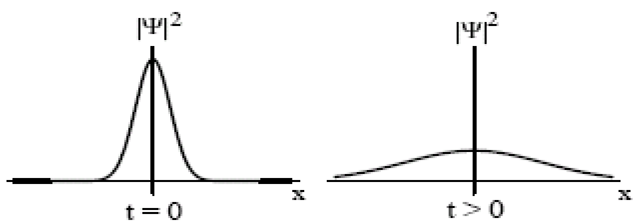
Does a Measured Spin State Smear Out into Other Spin States?

If you measure S_z of a spin-1/2 particle and collapse its wavefunction into the $\begin{pmatrix}1\\0\end{pmatrix}$ state, will it stay in that state forever, or will it evolve into states with non-zero $\langle S_x \rangle$ and $\langle S_y \rangle$ over time? Assume that the Hamiltonian governing the evolution of the particle after measurement has no spin dependence. Does the probability amplitude flow into other directions associated with the incompatible operators S_y and S_x?

One might think the answer is yes because the free particle, when prepared in a minimum uncertainty state "spreads out" with time. A particle, when initially prepared in a minimum uncertainty state with $\sigma_x \sigma_{p_x} = \hbar/2$, has the form $\Psi(x,0) = \left(\frac{2a}{\pi}\right)^{1/4} Exp\left[-ax^2\right]$. A free particle has a Hamiltonian $H = p^2/2m$, and obeys the time-dependent Schrödinger equation $i\hbar\partial\Psi/\partial t = H\Psi$. As time evolves the initial wavefunction spreads out in space and becomes $\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{Exp[-ax^2/(1+2i\hbar at/m)]}{\sqrt{1+2i\hbar at/m}}$. In pictures, it does this:



What happens if we do the same thing with a spin-1/2 particle and measure the z-component of spin? Suppose the outcome of the S_z measurement is $+\hbar/2$, and the wavefunction collapses into the $\begin{pmatrix}1\\0\end{pmatrix}$ state. The Hamiltonian once again is $H = p^2/2m$ and it is assumed that there is no magnetic field present that can interact with the magnetic moment associated with the spin. The spin state of the particle will remain in the $\begin{pmatrix}1\\0\end{pmatrix}$ state indefinitely. Why is it so different from the free-particle state that "spreads out"?

The answer lies in the issue of whether the observable in question commutes with the Hamiltonian or not. In general:

$$\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar}\langle [H,Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle \qquad \text{Griffiths [3.71]}$$

In the case of S_z one has $[H, S_z] = 0$ and $\partial S_z/\partial t = 0$ (the operator has no explicit time dependence), so $\frac{d}{dt}\langle S_z \rangle = 0$, and the spin remains in the same state indefinitely.

On the other hand, for the free particle, consider the operator x^2 (rather than x), so $\frac{d}{dt}\langle x^2\rangle = \frac{i}{\hbar}\langle [H, x^2]\rangle + \left\langle \frac{\partial x^2}{\partial t}\right\rangle$. Now the commutator is non-zero, $[H, x^2] = \frac{1}{2m}[p^2, x^2] = \frac{i\hbar}{2m}\left(x\frac{d}{dx} - 1\right)$. When this acts on the Gaussian wave packet it will have a non-zero expectation value, leading to a time-rate-of-change for $\langle x^2\rangle$, and a spreading of the wave packet. The exact answer for the Gaussian is [Griffiths Prob. 2.22] $\langle x^2\rangle = \frac{1+(2\hbar at/m)^2}{4a}$, showing that the particle undergoes a root-mean-square "displacement" that increases linearly with time.