Physics 401 - Homework #4

1) **Particle-in-a-box (three points each).** Consider a particle of mass (m) confined in a one-dimensional box between x = 0 and x = L. At t = 0 the state of the system is given by a square function:

$$\psi(x) = \sqrt{2/L}\ , \quad \frac{L}{4} < x < \frac{3L}{4}$$
$$\psi(x) = 0 \text{ , otherwise}$$

a) We wish to write this wavefunction as a sum of energy eigenstates:

$$\psi(x) = \sum_n a_n \varphi_n(x)$$

Calculate the correct set of expansion coefficients $\{a_n\}$ for this expression.

b) Suppose that at t = 0 we decide to measure the energy of the system. What is the probability that the result of the measurement will be $E = \dfrac{9\hbar^2\pi^2}{2mL^2}$?

c) Suppose that we perform the energy measurement, and that the result of the measurement is, in fact, $E = \dfrac{9\hbar^2\pi^2}{2mL^2}$. What will be the new wavefunction for the system after this energy measurement?

d) Now we let the new wavefunction evolve in time. What will be the fully time-dependent wavefunction, $\Psi(x,t)$?

e) After the energy measurement, will the position probability distribution for the system change as a function of time, or will it be constant in time? Explain your answer.

f) Suppose we wait 30 seconds, and then we measure the energy again. What is the probability that we get the same energy measurement, $E = \dfrac{9\hbar^2\pi^2}{2mL^2}$?

g) Suppose that we start over with the original wavefunction, shown at the top of the page, and this time we **do not** measure the energy. Instead we allow the system to evolve in time undisturbed. Will the position probability function depend on time in this case? Answer yes or no, and explain your answer.

2) **A symmetric box (three points each**). We have been studying the particle-in-a-box system with the x-coordinate measured from the left side of the box (so that the walls of the box are located at x = 0 and x = L). But some people prefer to have the walls of the box centered on the origin at x = -:L/2 and x = +L/2. This makes the potential function symmetric about x = 0. In this case, the stationary states (energy eigenfunctions) appear as:

$$\varphi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{L}\right), & n = odd \\ \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), & n = even \end{cases}.$$

(Note that these expressions for the $\varphi_n(x)$ are only true inside the box.... outside the box, the eigenfunctions are zero.)

a) Show that these states satisfy the boundary conditions for the particle-in-a-box system.

b) Show that these states satisfy the energy eigenvalue equation (the time-independent Schroedinger equation).

c) What are the energy eigenvalues?

d) These states satisfy an orthonormality condition:

$$\int \varphi_n^*(x)\varphi_m dx = \delta_{nm}$$

Show that this is true for the particular case of n = 3 and m = 4.

e) The parity operator $\hat{P}$ is defined such that it reverses the x-coordinate of any function of x:

$$\hat{P}f(x) = f(-x)$$

Show that the stationary states as written above are eigenfunctions of parity.

f) What are the eigenvalues of parity for the n = even and n = odd cases?

3) **Momentum probability distribution for a square pulse wavefunction (three points each).**

a) Calculate the momentum space wavefunction $\phi(k)$ for this square pulse wavefunction from Homework #2:

$$\Psi(x, t = 0) = \begin{cases} \sqrt{\frac{1}{L}}, \frac{-L}{2} < x < \frac{L}{2} \\ 0, otherwise \end{cases}$$

Hint #1: think Fourier Transform. Hint #2: You've already done something very similar to this on Homework #2.

b) Plot the momentum probability distribution $P(k) = |\phi(k)|^2$ as a function of (k) for the wavefunction from part (a). You may either sketch it by hand, or use a computer plotting program.

c) Can you tell what the expectation value for momentum is just by looking at P(k)? The answer is yes... but how can you tell?

d) Look at the width of the P(k) distribution from part (c). Qualitatively speaking what happens to the width of the distribution if we make L very small? What happens if we make L very large? (No calculations are necessary to answer this question.)

The bottom line: This is an illustration of the uncertainty principle: if we create a wavefunction $\psi(x)$ which is highly localized in a small region of space (a small L), then the width of the momentum distribution is very large, and a measurement of momentum will give a wide variety of values. Conversely, if L is very large, so the wavefunction is very spread out in space, then the momentum distribution can be very narrow, and a measurement of momentum will give only a small range of values.