i.e., $\underset{\text{number of zeroes of } f(z) \text{ inside } C}{(N} - \underset{\ldots\text{poles}\ldots}{P)} = \frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz$

Proof of 1st equality of Eq. 7.8 from chapter 14 of Boas (problem 7.42 from chapter 14)

(a) Residue of $\boxed{F(z)} = f'(z)/f(z)$ at zero of $f(z)$ of order n (at $z = a$): we have expansion of f around $z = a$ of the form

$$f(z) = a_n (z-a)^{n} + a_{n+1}(z-a)^{n+1} \ldots$$

order of zero

[no negative powers of $(z-a)$, since it's not a pole of $f(z)$]

Thus, $f'(z)$ [around $z = a$] $= n\, a_n (z-a)^{n-1} + (n+1) a_{n+1} \times (z-a)^n + \ldots$

differentiating above series term-by-term

so that we get

plugging in above series for f and f'

$$F(z)\ [\text{around } z = a] = \frac{[n\, a_n (z-a)^{n-1} + (n+1) a_{n+1} (z-a)^n + \ldots]}{[a_n (z-a)^n + a_{n+1}(z-a)^{n+1} + \ldots]}$$

dividing each term in numerator & denominator by $(z-a)^n$

$$= \frac{[n\, a/(z-a) + (n+1) a_{n+1} + \ldots]}{[a_n + a_{n+1}(z-a)^1 + \ldots]}$$

as leading term

Due to $1/(z-a)$ as leading term in numerator (vs. constant in denominator), there is pole for $F(z)$ at $z = a$ (which was expected anyway from start, since F has f in denominator)

So, residue of F (at $z = a$) $= \lim_{z \to a} (z-a) F(z)$

plug above series

$= [n\, a_n + 0 + \ldots] / [a_n + 0 + \ldots]$

$= \boxed{n}$

$\boxed{(b)}$ Similarly, residue of F at pole of $f(z)$ at $z = b$ of order p: we have the Laurent series:

$$f(z)\,[\text{around } z = b] = a_0 + a_1(z-b) + a_2(z-b)^2 + \dots$$
$$+ \frac{b_1}{z-b} + \dots + \frac{b_p}{(z-b)^p} \quad \left(\text{no } b_{p+1} \text{ onwards}\right)$$

so that
$$f'(z)\,[\text{around } z = b] = a_1 + 2a_2(z-b)^1 + \dots$$
$$- \frac{b_1}{(z-b)^2} + \dots \frac{(-p)\,b_p}{(z-b)^{p+1}}$$

Thus $F(z)\,[\text{around } z = b] =$

$$\frac{\left[a_1 + 2a_2(z-b) + \dots - b_1/(z-b)^2 + \dots \dfrac{(-p)\,b_p}{(z-b)^{p+1}}\right]}{\left[a_0 + a_1(z-b) + \dots + b_1/(z-b) + \dots + b_p/(z-b)^p\right]}$$

multiply numerator & denominator by $(z-b)^p$ (and change order of terms)

$$= \frac{\left[-\dfrac{p\,b_p}{(z-b)} + \dots - b_1(z-b)^{p-2} + a_1(z-b)^p + 2a_2(z-b)^{p+1} + \dots\right]}{\left[b_p + \dots b_1(z-b)^{p-1} + a_0(z-b)^p + a_1(z-b)^{p+1} + \dots\right]}$$

Clearly, there is a pole in $F(z)$ at $z = b$, with residue $= \lim_{z \to b} (z-b)\,F(z)$

$$= \left[-p\,b_p + 0 + \dots\right] / \left[b_p + 0 + \dots\right]$$

$$= \boxed{-p}$$

Summing over all zeroes & poles of f, we get 1st equality of Eq. 7.8