Note for Phys 373 (Fall 2015) by Dr. Agashe

$\boxed{\text{Claim}}$: the solution of differential equation

$$y'' + \left(\frac{1-2a}{x}\right) y' + \left[(b c x^{c-1})^2 + \frac{(a^2 - p^2 c^2)}{x^2}\right] y = 0 \quad \ldots (1)$$

is given by

$$y = x^a Z_p(b x^c) \quad \ldots (2)$$

where Z_p stands for J_p (Bessel function of 1st kind) or N_p / Y_p (Neumann/Weber function) of order p and a, b, c, p are constants.

$\boxed{\text{Proof}}$ Make the substitution/change of variables

$$y = x^a u(z), \text{ with } z = b x^c \quad \ldots (3)$$

and find equation to be satisfied by $u(z)$ such that y in (3) is solution of (1).

[Warning: the following is a tedious, but nonetheless fairly straightforward exercise in using product/chain rule of differentiation!]

Differentiating (3), we get

$$y' = a x^{a-1} u + x^a \underbrace{\frac{d u(z)}{dz} \frac{dz}{dx}}_{\text{i.e., effectively } \frac{d\, u[z(x)]}{dx}}$$

$\boxed{\text{1st}}$ term 2nd ...

$$= (a x^{a-1} u) + (x^a u' b c x^{c-1}) \ldots (4)$$

[we use the notation $\boxed{u'}$ for $\frac{du(z)}{dz}$, just like y' for $\frac{dy(x)}{dx}$]

Differentiating (4) gives

$$y'' = \left\{ a(a-1)x^{a-2}u + a x^{a-1} \overbrace{\boxed{u'}\underbrace{bc\,x^{c-1}}_{dz/dx}}^{\frac{d}{dx}u[z(x)] \text{ as before}} \right.$$

1st term in (4)

combine

$$+ \; bc\,(a+c-1)\,x^{a+c-2}\boxed{u'}$$

$$+ \; bc\,x^{a+c-1}\underbrace{u''\underbrace{bc\,x^{c-1}}_{dz/dx}}_{d/dx \text{ of } u'}$$

2nd term in (4)

simplifying

$$= (x^{a+c-2})(bc)\boxed{u'}(2a+c-1) + a(a-1)x^{a-2}u$$

1st term, 2nd...

$$+ \; b^2c^2x^{a+2c-2}u'' \qquad \ldots (5)$$

3rd...

If y in (3) is to satisfy (1), then we must have [plugging (5), (4) & (3) in (1)] that

$\to y''$, $\to y'$, $\to y$

1st of 5, 2nd of (4)

$$b^2c^2(x^{a+2c-2})(u'') + (u')\,bc\,x^{a+c-2}\left[\left(\frac{1-2a}{x}\right)x + (2a+c-1)\right]$$

from 3rd of (5)

$$+ \; u\,x^{a-2}\left\{a(a-1) + \left(\frac{1-2a}{x}\right)ax + \left[c^2(bx^c)^2 + a^2 - p^2c^2\right]\right\} = 0 \quad \ldots (6)$$

2nd of (5) — 1st of (4) — (3) plugged in last 2 terms of (1)

Cancelling x^a (present in all terms) and multiplying (6) by x^2 (and simplifying) gives

$$(b x^c)^2 c^2 (u'') + (u')(b x^c) c^2 + u\, c^2 [(b x^c)^2 - p^2] = 0$$

from 1^{st} term of (6) from [...] in 2^{nd} term of (6) from {...} in 3^{rd} term of (6)

Cancelling c^2 and using $z = b x^c$, we finally get

$$z^2 u'' + z u' + u(z^2 - p^2) = 0$$

i.e., solution for $u(z)$ is $Z_p(z)$ [Bessel function]

so that (3) becomes $y = x^a Z_p(b x^c)$, i.e.,

the claimed solution in (2) above to original equation (1)!