PHYS 373 (Fall 2015):
Mathematical Methods for Physics II

Summary of topics/formulae for Final exam

Chapter 7 of Boas (Fourier Series and Transforms)

1. General Fourier series: a function $f(x)$ with period $2l$ can be expanded as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_1^\infty \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right) && (1) \\ &= \sum_{-\infty}^{\infty} c_n e^{in\pi x/l} && (2) \end{aligned}$$

where the Fourier coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx && (3) \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx && (4) \\ c_n &= \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx && (5) \end{aligned}$$

2. Special Fourier series: we have

$$\text{if } f(x) \text{ is odd,} \quad \begin{cases} b_n = \frac{2}{l}\int_0^\infty f(x)\sin\frac{n\pi x}{l}dx \\ a_n = 0 \end{cases} \qquad (6)$$

$$\text{if } f(x) \text{ is even,} \quad \begin{cases} a_n = \frac{2}{l}\int_0^\infty f(x)\cos\frac{n\pi x}{l}dx \\ b_n = 0 \end{cases} \qquad (7)$$

3. Paserval's theorem for Fourier series:

$$\begin{aligned} \text{The average of } |f(x)|^2 \text{ (over a period)} &= \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + + \frac{1}{2}\sum_1^\infty b_n^2 && (8) \\ &= \sum_{-\infty}^{\infty} |c_n|^2 && (9) \end{aligned}$$

4. General Fourier transform

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}d\alpha \qquad (10)$$

where

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx \qquad (11)$$

5. Special Fourier transform: for an odd function, we have

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\alpha) \sin \alpha x \, d\alpha \tag{12}$$

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(x) \sin \alpha x \, dx \tag{13}$$

Similarly, for an even function:

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\alpha) \cos \alpha x \, d\alpha \tag{14}$$

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(x) \cos \alpha x \, dx \tag{15}$$

$$\tag{16}$$

6. Paserval's theorem for Fourier transform:

$$\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} \frac{1}{2\pi} |f(x)|^2 dx \tag{17}$$

Chapter 3 of Boas (Linear Algebra)

1. n-dimensional vector-space:

$$\mathbf{A}.\mathbf{B} \text{ (inner product)} = \sum_1^n A_i B_i \tag{18}$$

$$A \text{ (norm)} = \sqrt{\mathbf{A}.\mathbf{A}} \tag{19}$$

$$\mathbf{A} \text{ and } \mathbf{B} \text{ are orthogonal} \quad \text{if} \quad \mathbf{A}.\mathbf{B} = 0 \tag{20}$$

2. vector-space of functions on $a \le x \le b$:

$$\text{Inner product of } A(x) \text{ and } B(x) = \int_a^b A^*(x) B(x) dx \tag{21}$$

$$\text{Norm of } A(x) = \sqrt{\int_a^b A^*(x) A(x) dx} \tag{22}$$

$$A(x) \text{ and } B(x) \text{ are orthogonal} \quad \text{if} \quad \int_a^b A^*(x) B(x) dx = 0 \tag{23}$$

3. Gram-Schmidt method for making a basis ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$...) orthonormal:

$$\mathbf{e}_1 = \frac{\mathbf{A}}{A} \tag{24}$$

$$\mathbf{e}_2 = \text{normalized} \left[\mathbf{B} - (\mathbf{e}_1.\mathbf{B})\, \mathbf{e}_1\right] \tag{25}$$

$$\mathbf{e}_3 = \text{normalized} \left[\mathbf{c} - (\mathbf{e}_1.\mathbf{C})\, \mathbf{e}_1 - (\mathbf{e}_2.\mathbf{C})\, \mathbf{e}_2\right] \tag{26}$$

Chapter 12 of Boas (Series Solutions of Differential Equations)

1. Series method for solving (linear) ordinary differential equations (ODE): assume a solution of the form (with a's being coefficients *to be found*)

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{27}$$

giving

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{28}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \tag{29}$$

Plug the above series into each term of the ODE. Find the *total* coefficient of each power of x on each side of ODE and equate them (again, for each power of x). This will give the higher a coefficients in terms of lower ones.

2. Legendre's equation:

$$\left(1-x^2\right) y'' - 2xy' + l(l+1)y = 0 \tag{30}$$

has a solutions for each integer l (chosen to be non-negative) which is called the Legendre polynomial, $P_L(x)$ defined with

$$P_l(1) = 1 \tag{31}$$

For example,

$$P_0(x) = 1,\ P_1(x) = x,\ \ P_2(x) = \tfrac{1}{2}\left(3x^2-1\right),\ P_3(x) = \tfrac{1}{2}\left(5x^3-3x\right) ... \tag{32}$$

3. Rodrigues' formula for Legendre polynomials

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left(x^2-1\right)^l \tag{33}$$

4. Generating function for Legendre polynomials:

$$\Phi(x,h) = \left(1-2xh+h^2\right)^{-1/2},\ |h|<1 \tag{34}$$

$$= \sum_{l=0}^{\infty} h^l P_l(x) \tag{35}$$

5. Recursion relations for Legendre polynomials:

$$\begin{aligned}
lP_l(x) &= (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x), && (36)\\
xP_l'(x) - P_{l-1}'(x) &= lP_l(x), && (37)\\
P_l'(x) - xP_{l-1}'(x) &= lP_{l-1}(x), && (38)\\
\left(1-x^2\right)P_l'(x) &= lP_{l-1}(x) - lxP_l(x), && (39)\\
(2l+1)P_l(x) &= P_{l+1}'(x) - P_{l-1}'(x), && (40)\\
\left(1-x^2\right)P_{l-1}'(x) &= lxP_{l-1}(x) - lP_l(x) && (41)
\end{aligned}$$

6. Orthogonality of Legendre polynomials:

$$\int_{-1}^{1} P_l(x)P_m(x)dx = 0, \quad \text{unless } l = m \tag{42}$$

7. Normalization of Legendre polynomials:

$$\int_{-1}^{1} \left[P_l(x)\right]^2 = \frac{2}{2l+1} \tag{43}$$

$$\tag{44}$$

8. A function defined over the interval $(-1,1)$ can be expanded in a Legendre series

$$f(x) = \sum_{l=0}^{\infty} c_l P_l(x) \tag{45}$$

$$\tag{46}$$

where

$$c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_l(x)dx \tag{47}$$

9. Associated Legendre functions:

$$P_l^m(x) = \left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_l(x) \tag{48}$$

satisfy the equation

$$\left(1-x^2\right)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0 \tag{49}$$

For *each* m, they a set of orthogonal functions on $(-1,1)$, with normalization:

$$\int_{-1}^{1}\left[P_l^m(x)\right]^2 dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!} \tag{50}$$

10. Bessel equation

$$x^2 y'' + xy' + \left(x^2 - p^2\right) y = 0 \tag{51}$$

has solutions (Bessel functions):

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p} \tag{52}$$

and

$$N_p(x) = \frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin \pi p} \tag{53}$$

11. Asymptotic values:

$$\begin{aligned} J_0(0) &= 1 && (54) \\ J_{n\neq 0}(0) &= 0 && (55) \\ J_{n=0,1,2\ldots}(\infty) &= 0 && (56) \end{aligned}$$

12. Recursion relations for Bessel functions

$$\begin{aligned} \frac{d}{dx}\left[x^p J_p(x)\right] &= x^p J_{p-1}(x) && (57) \\ \frac{d}{dx}\left[x^{-p} J_p(x)\right] &= -x^{-p} J_{p+1}(x) && (58) \\ J_{p-1}(x) + J_{p+1}(x) &= \frac{2p}{x} J_p(x) && (59) \\ J_{p-1}(x) - J_{p+1}(x) &= 2J_p'(x) && (60) \\ J_p'(x) &= -\frac{p}{x} J_p(x) + J_{p-1}(x) = \frac{p}{x} J_p(x) - J_{p+1}(x) && (61) \end{aligned}$$

13. Other equations with Bessel function solutions

$$y'' + \frac{1-2a}{x} y' + \left[\left(bcx^{c-1}\right)^2 + \frac{a^2 - p^2 c^2}{x^2}\right] y = 0 \tag{62}$$

has the solution

$$y = x^a Z_p\left(bx^c\right), \text{ where } Z = J,\ N \tag{63}$$

and

$$y = J_p(Kx) \text{ and } N_p(Kx) \tag{64}$$

satisfy the equation

$$x\left(xy'\right)' + \left(K^2 x^2 - p^2\right) y = 0 \tag{65}$$

14. Orthogonality of Bessel functions:

$$\int_0^1 xJ_p(\alpha x)J_p(\beta x) \quad = \quad \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \frac{1}{2}J_{p+1}^2(\alpha) = \frac{1}{2}J_{p-1}^2(\alpha) = \frac{1}{2}J_p'^{\,2}(\alpha) & \text{if } \alpha = \beta \end{cases} \tag{66}$$

where α and β are zeroes of $J_p(x)$.

Chapter 13 of Boas (Partial Differential Equations)

1. Laplace equation in two-dimensional rectangular/Cartesian coordinates (for example, for steady-state temperature):

$$\frac{\partial^2}{\partial x^2}T(x,y) + \frac{\partial^2}{\partial y^2}T(x,y) \quad = \quad 0 \tag{67}$$

has basis functions (i.e., general solution is a suitable *combination* of these):

$$T(x,y) \quad = \quad \left\{ \begin{array}{c} e^{kx} \\ e^{-kx} \end{array} \right\} \left\{ \begin{array}{c} \sin ky \\ \cos ky \end{array} \right\} \text{ or } \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} \left\{ \begin{array}{c} e^{ky} \\ e^{-ky} \end{array} \right\} \tag{68}$$

2. Diffusion equation in one dimension

$$\frac{\partial^2 u}{\partial x^2} \quad = \quad \frac{1}{\alpha^2}\frac{\partial u}{\partial t} \tag{69}$$

has basis functions

$$u \quad = \quad e^{-k^2\alpha^2 t} \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} \tag{70}$$

3. Schroedinger equation in one dimension for a *free* particle (i.e., no potential):

$$-\frac{h^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = ih\frac{\partial \Psi}{\partial t} \tag{71}$$

has basis functions

$$\Psi \quad = \quad \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} e^{-iEt/h} \tag{72}$$

4. Wave equation in circular coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} \quad = \quad \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} \tag{73}$$

has basis functions:

$$z \quad = \quad \left\{ \begin{array}{c} J_n(Kr) \\ N_n(Kr) \end{array} \right\} \left\{ \begin{array}{c} \sin n\theta \\ \cos n\theta \end{array} \right\} \left\{ \begin{array}{c} \sin Kvt \\ \cos Kvt \end{array} \right\} \tag{74}$$

5. Laplace equation in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \tag{75}$$

has basis functions (where l is a non-negative integer, with $-l \leq m \leq +l$)

$$u = \left\{\begin{array}{c} r^l \\ r^{-l-1} \end{array}\right\} P_l^m(\cos\theta) \left\{\begin{array}{c} \sin m\phi \\ \cos m\phi \end{array}\right\} \tag{76}$$

Chapter 14 of Boas (Functions of a Complex Variable)

1. Basics of complex-valued functions of complex variable

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \tag{77}$$

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \tag{78}$$

2. If $f(z)$ is analytic in a region (i.e., has a unique derivative at every point), then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{79}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{80}$$

(Cauchy-Reimman conditions) and it's converse: if $u(x,y)$ and $v(x,y)$ satisfy these conditions, then $f(z) = u + iv$ is analytic.

3. If $f(z)$ is analytic in a region R, then it has derivatives of all orders at points inside R and thus it can be expanded in a Taylor series about any point z_0 in R. This power series converges *in*side circle C about z_0 that extends to the nearest singularity point (i.e., C *just* touches the boundary of R).

4. If $f(z) = u + iv$ is analytic in a region, then u and v satisfy (two-dimensional) Laplace's equation in the region. And, conversely, *any* function u (or v) satisfying Laplace's equation is the real (or imaginary) part of an analytic function $f(z)$.

5. Cauchy's theorem: if $f(z)$ is analytic inside and on a closed curve C, then

$$\int f(z)dz = 0, \text{ around } C \tag{81}$$

6. Cauchy's integral formula: if $f(z)$ is analytic inside and on a closed curve C, then

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}dz, \text{ around } C \tag{82}$$

where $z = a$ is a point *in*side C.

7. Laurent series: Let C_1 and C_2 be two circles with center at z_0. If $f(z)$ is an anlaytic function in the region R between $C_{1,2}$, then it can be expanded in a convergent series in R

$$f(z) = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + ... + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... \tag{83}$$

associated with which are the following definitions:

(i) If all the b's are zero, then $f(z)$ is analytic at $z = z_0$ (regular point);

(ii) If $b_n \neq 0$, but all the subsequent b's are zero, then $f(z)$ is said to have a *pole* of *order* n a $z = z_0$. If $n = 1$ here, then $f(z)$ has a simple pole at $z = z_0$;

(iii) If there are infinite number of b's which are different than zero, then $f(z)$ has an *essential* singularity at $z = z_0$;

(iv) The coefficient b_1 of $1/(z - z_0)$ is called the *residue* of $f(z)$ at $z = z_0$.

8. Residue theorem:

$$\int f(z)dz \text{ (around } C) = 2\pi i. \text{ (sum of residues of } f(z) \text{inside } C) \tag{84}$$

where we go *counter*-clockwise around C.

9. Methods of finding residues of $f(z)$:

(A) coefficient b_1 in Laurent series about $z = z_0$;

(B) Simple pole:

$$R(z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{85}$$

and if $f(z) = g(z)/h(z)$, then

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \text{ if } \begin{cases} \text{if } g(z_0) = \text{ finite const.} \\ h(z_0) = 0, \ h'(z_0) \neq 0 \end{cases} \tag{86}$$

(C) Multiple poles: multiply $f(z)$ by $(z - z_0)^m$, where m is an integer $\geq n$ (order of pole), differentiate the result $(m - 1)$ times, divide by $(m - 1)!$, and evaluate the resulting expression at $z = z_0$.

10. Definite integrals using residue theorem:

(i) Change of variables;

(ii) If $P(x)$ and $Q(x)$ are polynomials with degree of $Q \geq$ degree of $P+2$ and if Q has no real zeroes, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} = 2\pi i. \left(\text{sum of residues of } \frac{P(z)}{Q(z)} \text{ in upper half-plane}\right) \tag{87}$$

(iii) If $P(x)$ and $Q(x)$ are polynomials with degree of $Q \geq$ degree of $P+1$ and if Q has no real zeroes, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{imx} = 2\pi i. \left(\text{sum of residues of } \frac{P(z)}{Q(z)} e^{imz} \text{ in upper half-plane}\right) \tag{88}$$

where $m > 0$.

(iv) Poles on boundary:

$$\begin{aligned} \int f(z)dz \text{ (around } C) &= 2\pi i. (\text{sum of residues at simple poles inside } C+ \\ &\qquad \frac{1}{2} \text{ sum of residues of poles on the boundary}\Big) \end{aligned} \tag{89}$$

(v) Branch cuts: for integrals involvong *fractional* powers (or logarithm) of x (and thus z), we have to choose contour such that we stay on one branch of the fractional power (say, angle of z between 0 and 2π) so that the function is *single*-valued.

(vi) Argument principle:

$$N - P = \tfrac{1}{2\pi i} \int \tfrac{f'(z)}{f(z)} dz \text{ (around } C) = \frac{1}{2\pi} \Theta_C \tag{90}$$

where N and P are the number of zeroes and poles, respectively, of $f(z)$ inside C and Θ_C is the change in angle of $f(z)$ around C.

11. Nature of $f(Z)$ at $Z = \infty$: it is a pole of order 2 if $f(1/z)$ is the same at $z = 0$ etc.
12. Residue at infinity:

$$(\text{residue of } f(Z) \text{at } Z = \infty) = -\left(\text{residue of } \frac{1}{z^2} f\left(\frac{1}{z}\right) \text{ at } z = 0\right) \tag{91}$$