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# Experiment V: The AC Circuit, Impedance, and Applications to High and Low Pass Filters

## I. References

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Halliday, Resnick and Krane, *Physics*, Vol. 2, 4th Ed., Chapters 33

Purcell, *Electricity and Magnetism*, Chapter 4

## II. Equipment

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Digital Oscilloscope

Waveform Generator

Differential Amplifier

A variety of Resistors, Capacitors, and Inductors

Digital Multimeter

LCR meter

Capacitor box

## III. Introduction

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We have now been introduced to resistors, capacitors and inductors and investigated their DC behavior. Now, we will see how they behave in an AC circuit. We will see that the response of RL and RC circuits are frequency dependent, and we can exploit this to build a circuit that filters unwanted frequencies.

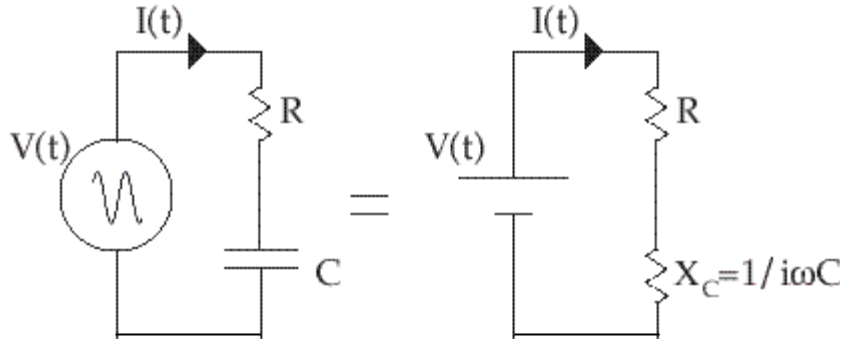
Filters are ubiquitous in electronics and have many uses. For example, small audio speakers are more efficient at reproducing high frequencies (such speakers are known as “tweeters”) and larger speakers are more efficient at reproducing lower frequencies (“woofers”). (*An aside: Why is this true? You will cover this in PHYS273 if you haven’t already.*) Your stereo or home theater system may contain filters to separate the high and low frequency components before channeling them to the appropriate speaker. Some sensitive electronic devices will contain a “bandstop” filter that will remove 60 cycle noise from home AC power delivery. Conversely, your cell phone will contain a “bandpass” filter, which selectively passes only the frequencies allocated to you, so you can hear your conversation, and not everyone else’s! Your cell phone “tunes” to a frequency band in much the same way as your radio tunes to a particular station.

We will start here with simple AC circuits that contain either an inductor or a capacitor, but not both. Along with a resistor, these components can be used to form a low pass or a high pass filter, respectively.

### A. Capacitive Impedance

Figure V-1 shows a time dependent voltage source  $V(t)=V_0\cos(\omega t)$  connected in series with a resistor  $R$  and a capacitor  $C$ . In order to find the current  $I(t)$  through the circuit, we can treat the capacitor as having a complex impedance, and then use Kirchhoff's laws to analyze the circuit. This AC circuit is "equivalent" (as far as finding the current is concerned) with a pseudo DC circuit with the same  $V(t)$ , but with the capacitor replaced by a resistor with a complex impedance given by the equation

$$X_c \equiv \frac{1}{i\omega C}.$$



**Figure V-1: DC equivalent of an AC circuit with a resistor and capacitor**

This is a DC circuit with a voltage source given by  $V(t) = V_0 e^{i\omega t}$  and an equivalent resistance

$$R_{eq} = R - i \frac{1}{\omega C} = Z e^{i\phi},$$

where

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = R \sqrt{1 + \left(\frac{1}{\omega RC}\right)^2} = R \sqrt{1 + \left(\frac{1}{\omega \tau}\right)^2}$$

and

$$\tan \phi = \frac{-1}{\omega RC} = \frac{-1}{\omega \tau}$$

with  $\tau = RC$ . The effective resistance of the circuit now has a magnitude and a phase angle  $\phi$ . The magnitude will allow us to determine the current through the circuit:

$$Z = R \sqrt{1 + \left(\frac{1}{\omega \tau}\right)^2} = R \sqrt{1 + \tan^2 \phi} = R \sec \phi = \frac{R}{\cos \phi}.$$

and the current  $I = V/R_{eq}$  is

$$I(t) = V(t) / R_{eq} = \frac{V_0}{R} \cos \phi e^{i(\omega t - \phi)}.$$

Of course, we always take the “real part” to get real voltages and currents. Note that the current through the circuit is phase-shifted from the input voltage by the angle  $\phi$ . We can now calculate the voltage drop across the resistor and the capacitor using the usual DC rule for “resistors”  $V=IR$ :

$$V_R = IR = V_0 \cos \phi e^{i(\omega t - \phi)}$$

$$V_C = IX_C = \frac{V_0}{R} \cos \phi e^{i(\omega t - \phi)} \frac{1}{i\omega C} = \frac{V_0}{i\omega RC} \cos \phi e^{i(\omega t - \phi)}.$$

We then use the fact that  $\frac{1}{i\omega RC} = i \tan \phi = \tan \phi e^{i\pi/2}$  to write

$$V_C = V_0 \tan \phi \cos \phi e^{i(\omega t - \phi + \pi/2)} = V_0 \sin \phi e^{i(\omega t - \phi + \pi/2)} = -V_0 \sin \phi \sin(\omega t - \phi),$$

where the last step is a result of keeping only the real part of  $V_C$ . The results are summarized in Table V-1 below. A few results of these basic equations are:

1. The current  $I$  and the voltage drop across the resistor  $V_R$  are in phase with each other, and out of phase with the input voltage  $V(t)$ .
2. The voltage drop across the capacitor  $V_C$  is out of phase by a quarter cycle ( $\pi/2$ )
3. The input voltage, current,  $V_R$ , and  $V_C$  all have the same frequency.

Input Voltage	$V = V_0 \cos(\omega t)$
Current	$I = I_0 \cos \phi \cos(\omega t - \phi)$
Voltage drop across R	$V_R = V_0 \cos \phi \cos(\omega t - \phi)$
Voltage drop across C	$V_C = -V_0 \sin \phi \sin(\omega t - \phi)$
Initial current amplitude	$I_0 \cos \phi \equiv (V_0/R) \cos \phi$
RC Decay time	$\tau \equiv RC$
Phase shift	$\tan \phi \equiv -1/\omega\tau$

**Table V-1: Properties of the current and various voltages in the RC circuit. Note that  $I_0 = V_0/R$ .**

Figure V-2 shows the 3 signals  $V(t)$  (input voltage),  $V_R(t)$ , and  $V_C(t)$  for the phase  $\phi = -0.6$  radians. We see that the voltage across the resistor leads (is ahead of in time) the input voltage, and has a decreased amplitude, whereas the voltage across the capacitor lags (is behind in time)

the input voltage. Notice also that  $V_C$  and  $V_R$  are out of phase by  $\pi/2$  (seen by the fact that when  $V_R$  peaks,  $V_C$  is at zero).

The “leading” and “lagging” properties can also be seen by writing  $\omega\tau - \phi = \omega(\tau - \phi/\omega)$  which allows you to calculate how much in time the signals differ according to the phase difference. In Figure V-2, note that  $V_R$  leads  $V(t)$  by a time difference  $\delta t = \phi/\omega$ .

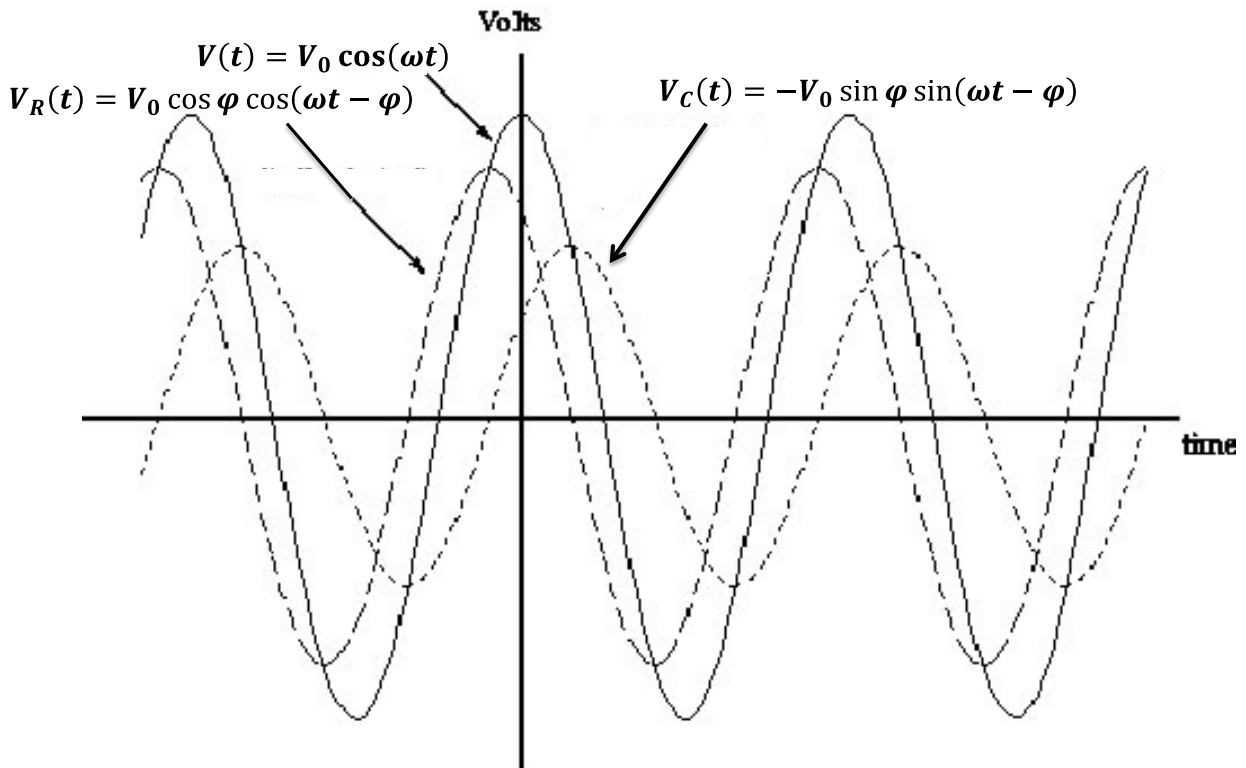


Figure V-2: Voltages in an AC RC circuit for the phase  $\phi = -0.6$  radians

## B. Inductive Impedance

We can also think of an inductor as having a complex impedance, but with the impedance given by

$$X_L = i\omega L \text{ and a time constant } \tau = L/R$$

So, unlike the capacitive impedance, the inductive impedance is directly proportional to the frequency. If you think about it, this should make sense intuitively. An inductor tends to “fight” change in current, by inducing a back emf, thus, the higher the rate of change, the more it “resists” current. A capacitor tends to fight a change in voltage by storing charge. At high

frequencies, the capacitor never fully charges, and therefore it doesn't have a high resistance. The impedance of a capacitor is inversely proportional to frequency.

In an inductive circuit, the voltage across the inductor will lead the input voltage, or emf. Put another way, the current (or equivalently, the voltage across the resistor) lags. One way to keep this straight in your mind is to remember ELI the ICE man, where E represents the input voltage or emf. The Emf Leads the current, I, in an inductive (L) circuit whereas the current (I) leads the Emf in a capacitive circuit.

In an RL series circuit, the total impedance is

$$Z = [R^2 + (\omega L)^2]^{1/2}$$

### III. Introduction to High Pass and Low Pass Filters

From the expressions for impedance, we can see that in the limit of a DC current (zero frequency), a capacitor will behave as an "open circuit". In other words, once the capacitor charges up, no more current can flow (infinite resistance). In the limit of very high frequency (approaching infinity), the same circuit will have nearly zero resistance, and will be have like a short circuit.

Conversely, we see that an inductor will behave like a short circuit for a DC current, and an open circuit as the frequency approaches infinity.

So, what is all of this good for? We can exploit this behavior to build filters. We have components whose impedance is frequency dependent. By Ohm's Law, this means that the ratio of the output voltage on the resistor to the input voltage (often called the "gain", G) is frequency dependent. Because these are "passive" components (not powered), the highest gain we will achieve is one.

For the capacitive circuit shown in Fig. V-1:

$$G = R/[R^2 + (1/\omega C)^2]^{1/2} = 1/[1 + 1/(\omega\tau)^2]^{1/2}, \text{ with } \tau = RC$$

Note that this becomes 1 at very high frequencies, and 0 for DC currents, as we know it should.

For an inductive (RL) circuit:

$$G = R/[R^2 + (\omega L)^2]^{1/2} = 1/[1 + (\omega\tau)^2]^{1/2}, \text{ with } \tau = L/R$$

Again, we can verify that this expression predicts the expected behavior in the limit of low and high frequencies.

## IV. Experiments

### Experiment Part A

In this part, you will measure the phase shift for an RC and RL circuit.

**A.1:** Set up an RC circuit by putting a 10 kΩ resistor and a 0.1 μF capacitor in series with the signal generator. Use a ~100 Hz sine wave as input and display the two signals  $V_o(t)$  and  $V_R(t)$  on the scope. (When setting up the circuit, you may wish to think a bit about the instrumentation amplifier. How can you set up the circuit to avoid using it? How could you set up the circuit to require its use?) Use the TTL signal as the external trigger so that the displayed signals always have a fixed time reference.

You should see the phase shift between the two signals as a shift in time, just as in Figure IV-2 above. Capture your waveform and paste into lab report. The phase shift  $\phi$  and the time shift  $\delta t$  are related by the fact that the signal oscillates in time with period (phase  $2\pi$ ) given by  $T=1/f$ . The phase angle is defined as the fraction of a full period corresponding to the time shift  $\delta t = \phi/\omega$ :

$$\frac{\phi}{2\pi} = \frac{\delta t}{T}$$

or

$$\phi = 2\pi \frac{\delta t}{T} = (2\pi f) \delta t = \omega \delta t .$$

From a measurement of the time difference  $\delta t$  between the zero crossings of the two signals, determine the phase difference  $\phi$  (**call this  $\phi_t$** ). Estimate the uncertainty in the phase shift from your estimate of the uncertainties in the measured time difference.

**A.2:** The amplitude of  $V_R$  will be less than (or equal to) the amplitude of  $V_o(t)$ , which is defined as  $V_o$ . The reduction in amplitude can be shown to have a simple dependence on the phase shift.

$$|V_R/V_o| = G = R/[R^2+(1/\omega C)^2]^{1/2} = |\cos \phi|$$

Measure  $V_R$  and  $V_o$ , and record them. Do your values for gain  $G$  agree? Convert this value of  $G$  into a phase shift, and **call it  $\phi_v$** . Compare your result for  $\phi_v$  and  $\phi_t$  using a statistical test. Which can give a more precise value for  $\phi$ , the measurement in part A.1 or the value extracted using the relation given above.

**A.3:** Equivalently, we can write

$$|V_C/V_R| = 1/\omega RC = 1/\omega\tau = \tan\phi$$

Thus, if we plot the amplitude  $|V_R/V_C|$  vs.  $\omega$  (Remember that  $\omega = 2\pi f$ !), the slope should be  $\tau$ . Please make this plot, evaluate  $\tau$  from the slope, and compare to the value of  $\tau$  calculated from  $R$

and C. Use frequencies from 50 to 500 Hz (multiply by  $2\pi$  to get  $\omega$ ). Take about seven data points. **Do not do error analysis for this section.** Be sure to notice that the phase shift can change sign.

**A.4:** Now let's look at the phase shift for an inductive circuit. For a 100 mH inductor and a resistor of 50 to 100  $\Omega$ , use a  $\sim 1$  kHz sine wave and determine the phase shift by measuring it both directly from the time shift you observe on your scope screen, and by measuring the ratio of the input voltage, and the voltage across the resistor. Compare your results using a statistical test.

## Experiment Part B

Now, you are to design two circuits, one which supplies a woofer, and the other that sends sounds to a tweeter. Pretend the transition between your woofer and your tweeter occurs between  $f = 2000$  and  $5000$  Hz. Suppose your speaker has a resistance of  $50 \Omega$ .

For a filter, the cutoff frequency  $f_0$  is defined as the point where the gain  $G$  is  $1/\sqrt{2}$ . Find an expression for this frequency for each of your filters.

Discuss this with your partner, or think about it carefully yourself if you don't. What are the values of the components you will use in your circuits?

Once you have designed your circuit, plot the predicted impedance vs. frequency and the gain vs. frequency in Excel from 1 kHz to 100 kHz.

Now hook up your high pass filter, using a variable resistor box set at  $50 \Omega$  as a stand-in for your speaker, which in this case is your output or "load", and put in an input sine wave of 5V amplitude. Use the variable capacitor set to the value required to create your high pass filter. Set channel 1 of your scope to read the input voltage from the signal generator, and set channel 2 to measure the voltage across your resistor or "speaker". Measure the frequency dependent gain from 10 Hz to 100 kHz at 20 points that are equally spaced in  $\log_{10}$  (Excel can be used to fill in these values.) Record the frequency, the gain, and the uncertainty in the gain. You may use the math mode on your scope to calculate the gain. Does this agree to within your model prediction plotted above? If not, how do they differ?

Now sweep through the frequencies again and note the value of the phase shift of the output versus the input voltage. Describe how the phase and the gain are correlated. What is the gain when the phase shift is zero? What are the phase shifts on either end of the frequency spectrum?