BELL'S INEQUALITY

We shall be *slightly* mathematical. The details of the math are not important, but there are a couple of pieces of the proof that will be important. The result of the proof will be that for any collection of objects with three different parameters, *A*, *B* and *C*:

The number of objects which have parameter A but not parameter B plus the number of objects which have parameter B but not parameter C is greater than or equal to the number of objects which have parameter A but not parameter C.

We can write this more compactly as:

Number(A, not B) + Number(B, not C) greater than or equal to Number(A, not C)

The relationship is called *Bell's inequality*.

In class I often make the students the collection of objects and choose the parameters to be:

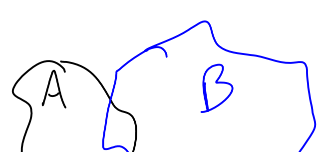
A: male **B**: height over 5' 8" (173 cm) **C**: blue eyes

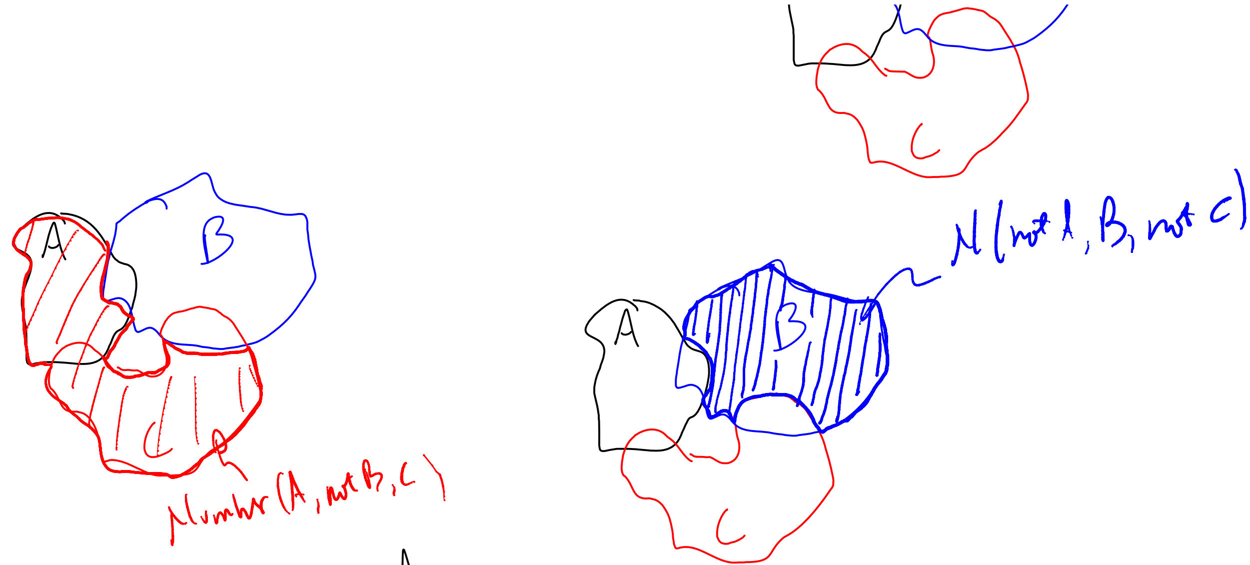
Now we are ready for the proof itself. First, I assert that:

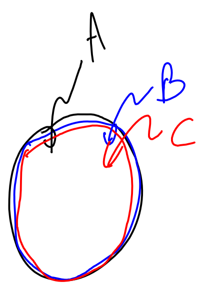
Number(A, not B, C) + Number(not A, B, not C) must be either 0 or a positive integer

or equivalently:

Number(A, not B, C) + Number(not A, B, not C) greater than or equal to 0







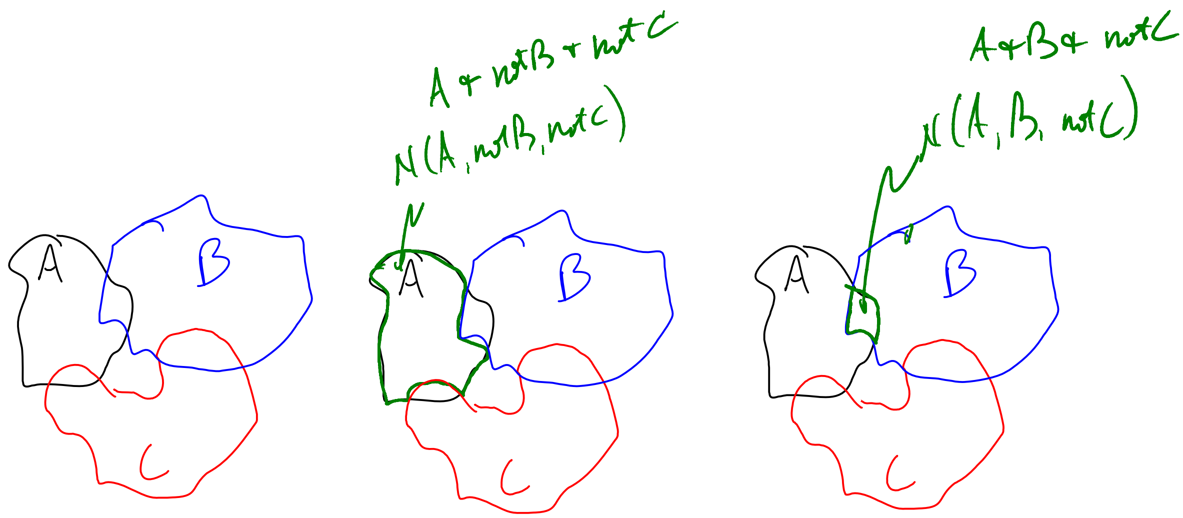
$A = B = C$

$\Rightarrow N(A, \text{not } B, C) = 0$

$N(\text{not } A, B, \text{not } C) = 0$

Now we add **Number(A, not B, not C) + Number(A, B, not C)** to the above expression. The left hand side is:

Number(A, not B, C) + **Number(A, not B, not C)** **+ Number(not A, B, not C) +**
Number(A, B, not C) >=
0 + **Number(A, not B, not C) + Number(A, B, not C)****=**
Number(A, not C)



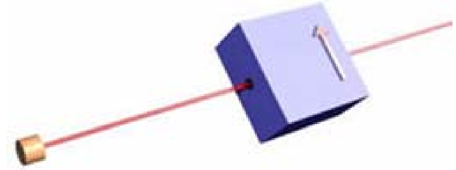
$N(A, \text{not } B, \text{not } C) + N(A, B, \text{not } C)$
$= N(A, \text{not } C)$

$N(A, \text{not } B)$ $\qquad$ $N(B, \text{not } C)$

[Number(A, not B, C) + Number(A, not B, not C)] + [Number(not A, B, not C) + Number(A, B, not C)] >=Number(A, not C)

$\therefore \quad N(A, \text{not } B) + N(B, \text{not } C) \geq N(A, \text{not } C)$

A: electrons are "spin-up" for an "up" being defined as straight up, which we will call an angle of zero degrees. **B**: electrons are "spin-up" for an orientation of 45 degrees. **C**: electrons are "spin-up" for an orientation of 90 degrees.

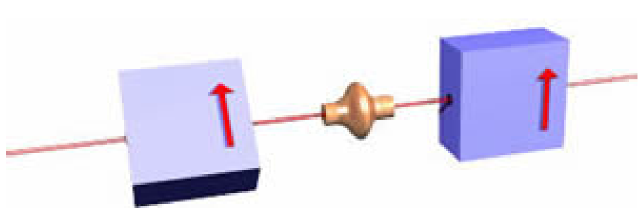


Then Bell's inequality will read:

Number(spin-up zero degrees, not spin-up 45 degrees) + Number(spin-up 45 degrees, not spin-up 90 degrees) >=
Number(spin-up zero degrees, not spin-up 90 degrees)

Note that the SG represents configuration: (spin-up zero degrees, not spin-up 45 degrees)

So we have "beaten" the Uncertainty Principle: we have determined whether or not the electron to the right is **spin-up zero degrees, not spin-up 45 degrees** by measuring its spin at zero degrees and its companion's spin at 45 degrees.
Now we can write the Bell inequality as:



Number(right spin-up zero degrees, left spin-up 45 degrees) + Number(right spin-up 45 degrees, left spin-up 90 degrees) >=
Number(right spin-up zero degrees, left spin-up 90 degrees)

Or let alpha (replacing 45 degrees) be an arbitrary angle, and beta (replaces 90 degrees) be an arbitrary angle:

Number(right spin-up zero degrees, left spin-up alpha degrees) + Number(right spin-up alpha degrees, left spin-up Beta degrees) >=
Number(right spin-up zero degrees, left spin-up beta degrees)

Using quantum mechanics, we know the probabilities for these events. The probability to detect right spin up at angle alpha and left spin up at angle beta is:

$$\frac{1}{2}\sin\left(\frac{\alpha-\beta}{2}\right)^2$$

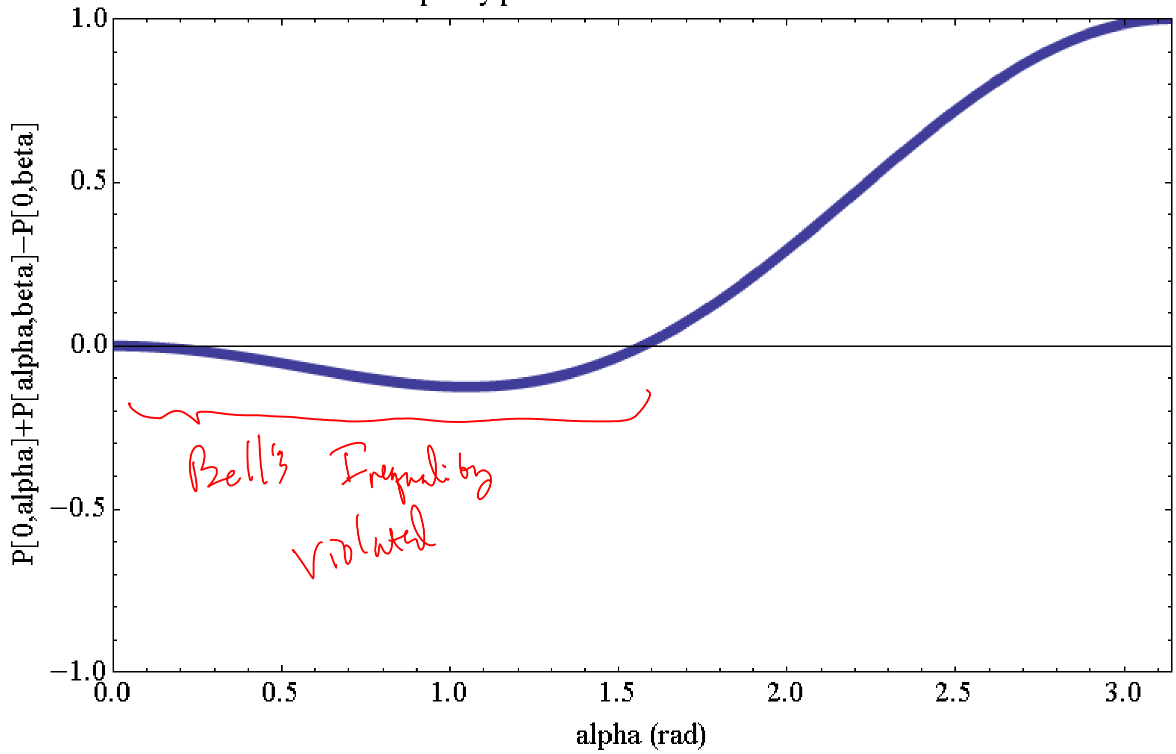
then Bell's inequality becomes:

$$\frac{1}{2}\sin\left(\frac{\alpha}{2}\right)^2+\frac{1}{2}\sin\left(\frac{\alpha-\beta}{2}\right)^2-\frac{1}{2}\sin\left(\frac{\beta}{2}\right)^2\geq 0$$

Choosing beta = 2 alpha for demonstration purposes, we obtain for Bell's inequality:

$$\sin\left(\frac{\alpha}{2}\right)^2-\frac{1}{2}\sin(\alpha)^2\geq 0$$

Quantum correlations in Stern Gurlach experiment
P[alpha,beta] means that right particle detected spin up with SG aligned to angle alpha,
left spin up detected with SG at angle beta
Let beta=2 alpha
Bell's Inequality predicts that this function should be >=0



In the last section we made two assumptions to derive Bell's inequality which here become:

- Logic is valid.
- Electrons have spin in a given direction even if we do not measure it.

Now we have added a third assumption in order to beat the Uncertainty Principle:

- No information can travel faster than the speed of light.

We will state these a little more succinctly as:

1. Logic is valid.
2. There is a reality separate from its observation
3. Locality.

You will recall the we discussed proofs by negation. The fact that our final form of Bell's inequality is experimentally violated indicates that at least one of the three assumptions we have made have been shown to be wrong.

You will also recall that earlier we pointed out that the theorem and its experimental tests have nothing to do with Quantum Mechanics. However, the fact that Quantum Mechanics correctly predicts the correlations that are experimentally observed indicates that the theory too violates at least one of the three assumptions.

Finally, as we stated, Bell's original proof was in terms of hidden variable theories. His assumptions were:

1. ✓ Logic is valid.
2. Hidden variables exist. Unknown
3. Hidden variables are local. X

Most people, including me, view the assumption of local hidden variables as very similar to the assumption of a local reality.