Physics for Scientists and Engineers

Chapter 12
Rotation of a Rigid Body

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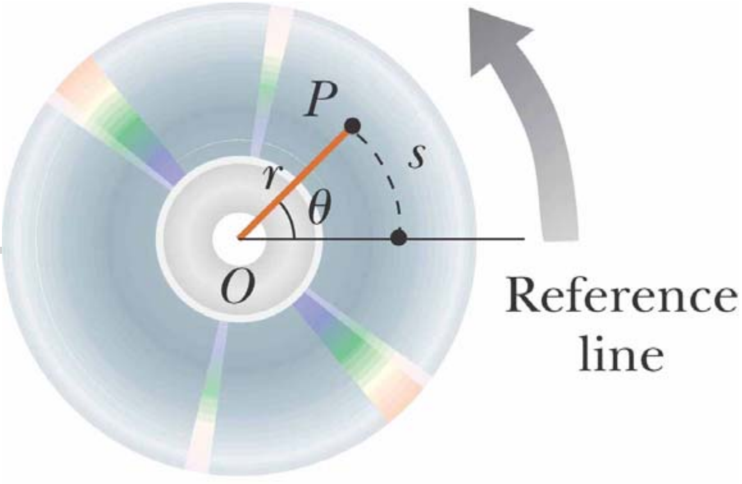
Rigid Body

- A *rigid object* is one that is non-deformable
  - The relative locations of all particles making up the object remain constant
  - All real objects are deformable to some extent
- The rigid object model is very useful in many situations where the deformation is negligible

Rotation of a Disk

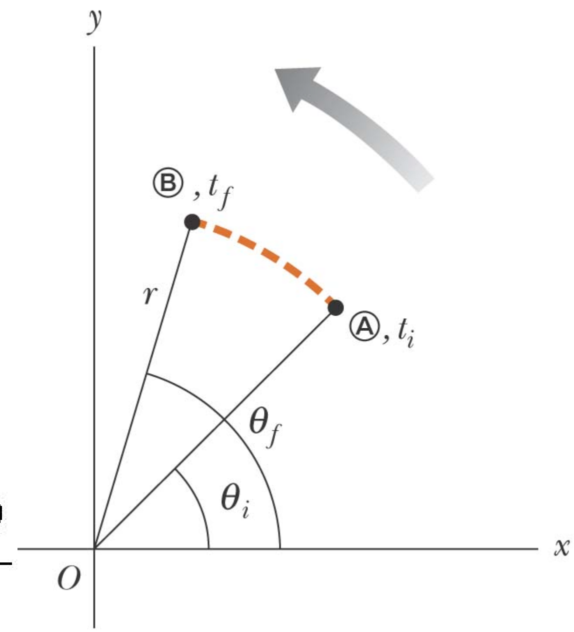


- Axis of rotation is through the center of the disk, *O*
- *Every particle* on the disk undergoes *the same circular motion* about the origin
- Polar coordinates are convenient to use to represent the position of *P*
  - *P* is located at (r, θ)
  - With θ measured in *radian*, the arc length and r are related by $s = \theta r$

Angular Speed and Acceleration

- Angular speed:

$$\bar{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \qquad \omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$



- Angular acceleration:

$$\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \qquad \alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

- Angular speed is *positive* (*negative*) for *counter-clockwise* (*clockwise*) rotation

Rotational Kinematics

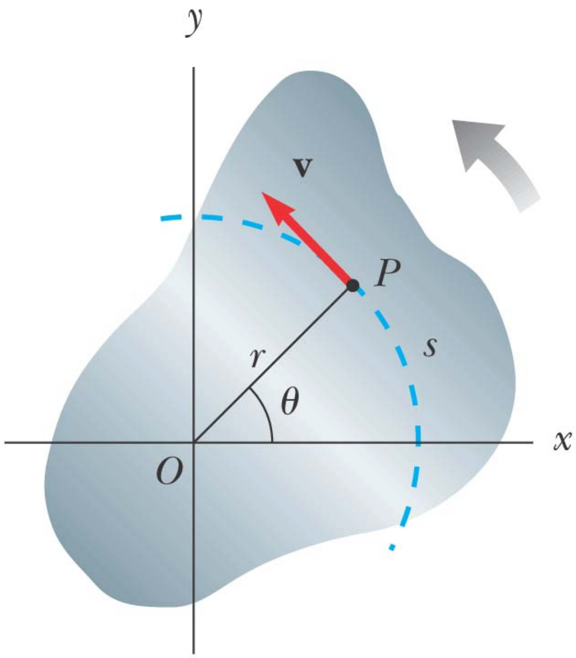
Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Linear and Angular Quantities

- The linear velocity is always *tangent* to the circular path
  - Called the *tangential velocity*

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$



- The *tangential acceleration* is the derivative of the tangential velocity

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

Center of Mass

- There is a special point in a system or object, called the *center of mass* (CM), that moves as if all of its mass ($M = \Sigma m_i$) is concentrated at that point
  - The system will move as if an external force were applied to a single particle of mass M located at the CM
- A general motion of an extended object can be represented as the sum of a linear motion of M at CM plus a rotation about the CM

Center of Mass, cont

- The coordinates of the CM are

$$x_{\mathrm{CM}} = \frac{\sum_i m_i x_i}{M} \quad y_{\mathrm{CM}} = \frac{\sum_i m_i y_i}{M} \quad z_{\mathrm{CM}} = \frac{\sum_i m_i z_i}{M}$$

- CM can be located by its position vector, $\mathbf{r}_{\mathrm{CM}}$

$$\mathbf{r}_{\mathrm{CM}} = \frac{\sum_i m_i \mathbf{r}_i}{M}$$

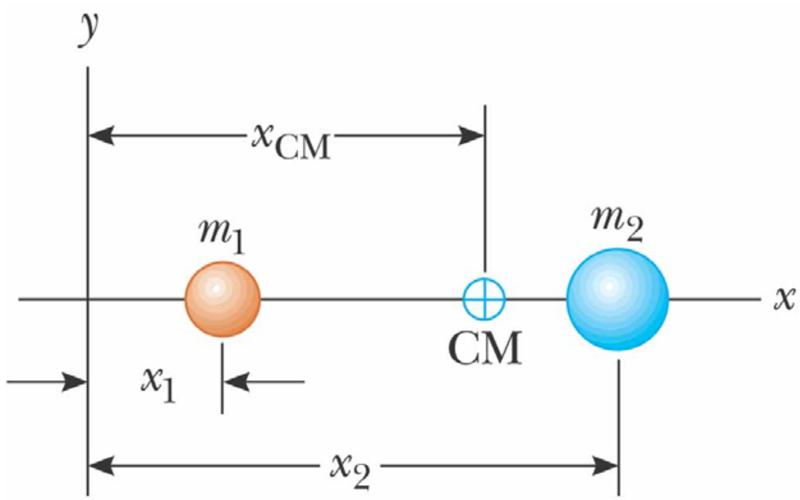
where $\mathbf{r}_i$ is the position of the ith particle:

$$\mathbf{r}_i = x_i \hat{\mathbf{i}} + y_i \hat{\mathbf{j}} + z_i \hat{\mathbf{k}}$$

Example 1: CM of Two Masses

Two masses, m_1 = 1.0 kg and m_2 = 2.0 kg are located on the x-axis at x_1 = 1.0 m and x_2 = 4.0 m, respectively.

Find the CM of the system.



$$\vec{r}_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}\hat{i}$$

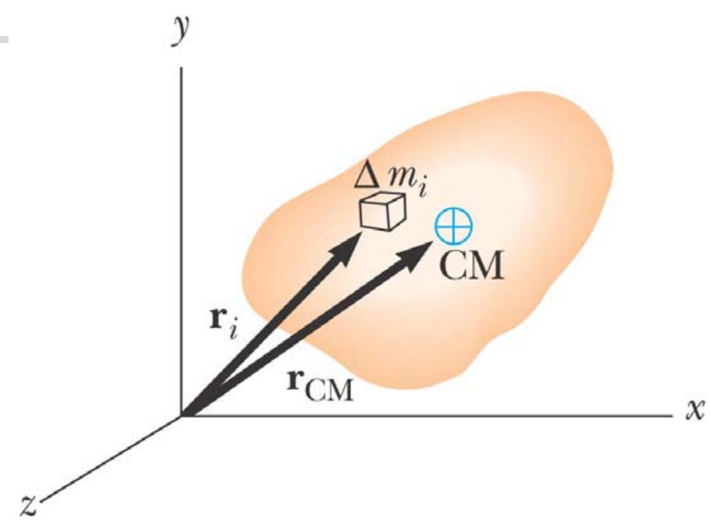
$$= \frac{(1.0\,\text{kg})(1.0\,\text{m}) + (2.0\,\text{kg})(4.0\,\text{kg})}{1.0\,\text{kg} + 2.0\,\text{kg}}\hat{i}$$

$$= 3.0\,\text{m}\,\hat{i}$$

- The CM is on the x-axis
- The CM is closer to the particle with the larger mass

CM of Extended Object

- An extended object can be considered a distribution of small mass elements, Δm



- The coordinates of the CM are

$$x_{CM} = \frac{1}{M}\int x dm, \;\; y_{CM} = \frac{1}{M}\int y dm, \;\; z_{CM} = \frac{1}{M}\int z dm$$

  - Vector position of the CM: $\mathbf{r}_{\mathrm{CM}} = \frac{1}{M}\int \mathbf{r}\, dm$

- The CM of any symmetrical object lies on an axis of symmetry and on any plane of symmetry

Notes on Various Densities

- *Volume* mass density, mass per unit volume:

$$\rho = m / V, \;\; dm = \rho dV$$

- *Surface* mass density, mass per unit area of a sheet of uniform thickness *t* :

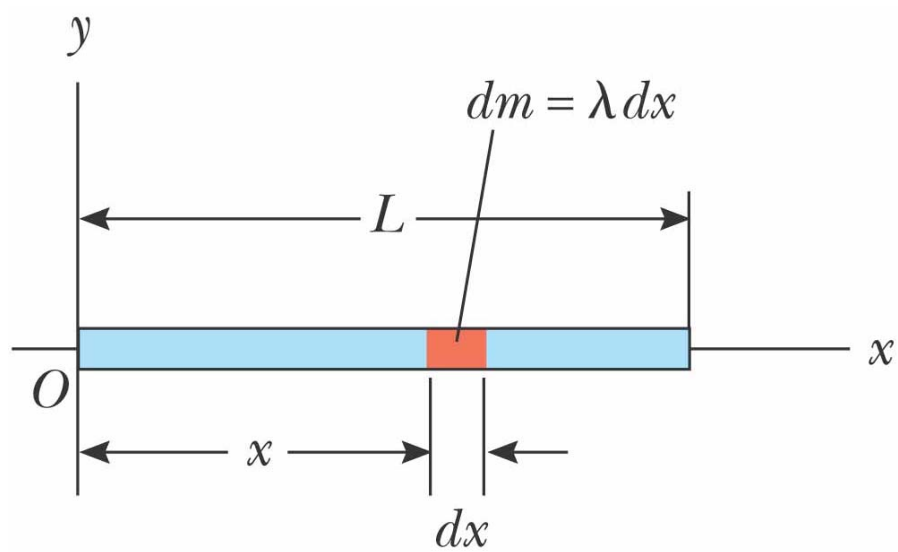
$$\sigma = m / A = \rho t, \;\; dm = \sigma dA$$

- *Line* mass density, mass per unit length of a rod of uniform cross-sectional area *A*:

$$\lambda = m / L = \rho A, \;\; dm = \lambda dx$$

Example 2: CM of a Rod

Find the center of mass of a rod of mass *M* and length *L*.



Let λ be the linear density.

Then,

$$dm = \lambda dx$$

$$M = \int dm = \lambda L$$

The location of the CM is on the x-axis:

$$y_{\text{CM}} = z_{\text{CM}} = 0,$$

$$x_{CM} = \frac{1}{M}\int x dm = \frac{1}{M}\int \lambda x dx = \frac{\lambda L^2}{2M}$$

$$= L/2$$

- The rod is symmetric with respect to $x = L/2$

Motion of a System of Particles

- The velocity of the CM of the system of particles is

$$\mathbf{v}_{\mathbf{CM}} = \frac{d\mathbf{r}_{\mathbf{CM}}}{dt} = \frac{1}{M}\sum_i m_i \mathbf{v}_i$$

  - The momentum can be expressed as $M\mathbf{v}_{\mathbf{CM}} = \sum_i m_i \mathbf{v}_i = \mathbf{p}_{\text{tot}}$

- The acceleration of the CM can be found as

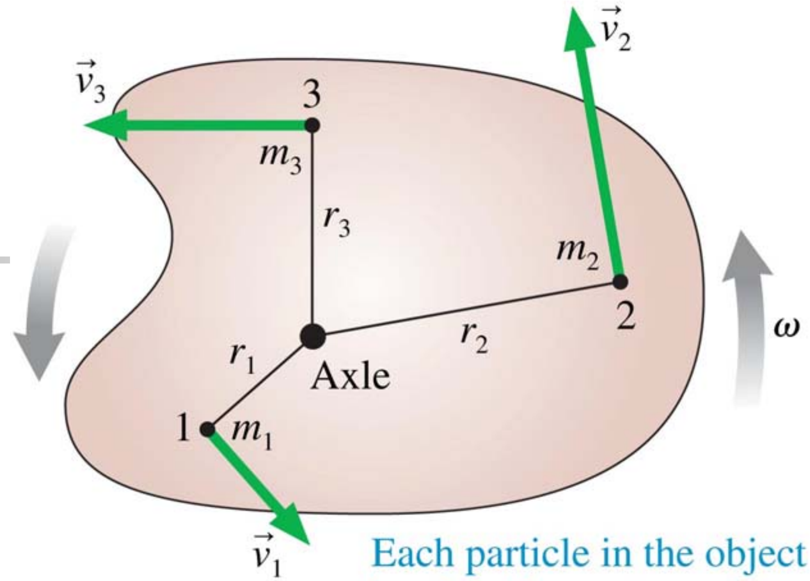
$$\mathbf{a}_{\text{CM}} = \frac{d\mathbf{v}_{\text{CM}}}{dt} = \frac{1}{M}\sum_i m_i \mathbf{a}_i$$

- The acceleration can be related to a force

$$M\mathbf{a}_{\text{CM}} = \sum_i F_i$$

Rotational Energy

- The object's rotational kinetic energy is the sum of kinetic energies of each of the particles:



Each particle in the object has kinetic energy as the object rotates.

$$K_{rot} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \cdots$$

$$= \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2}I\omega^2$$

- The quantity $\sum_i m_i r_i^2$ is called the *moment of inertia*

Moment of Inertia

- Moment of inertia is defined as $I = \sum_i m_i r_i^2$
  - Dimensions of I are ML^2 and its SI units are $kg{\cdot}m^2$
- For an extended object, we replace m_i with small mass element Δm_i and take the limit $\Delta m_i \to 0$:

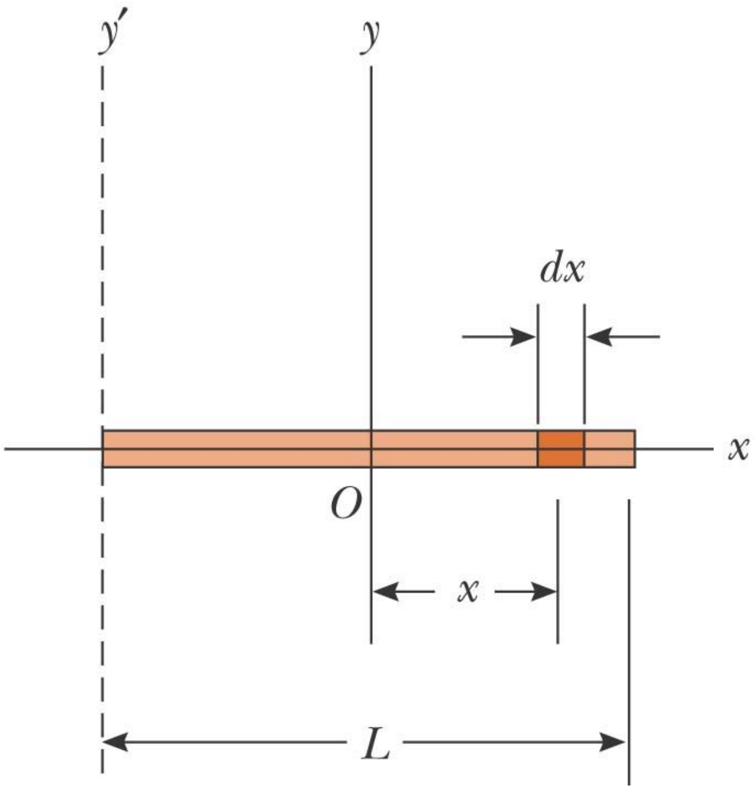
$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

- Expressing the mass element as $dm = \rho\, dV$

$$I = \int \rho r^2 dV$$

Example 3: I of a Rod

Uniform rigid rod
about its center



- The shaded area has a mass

$$dm = \lambda\, dx = (M/L)\, dx$$

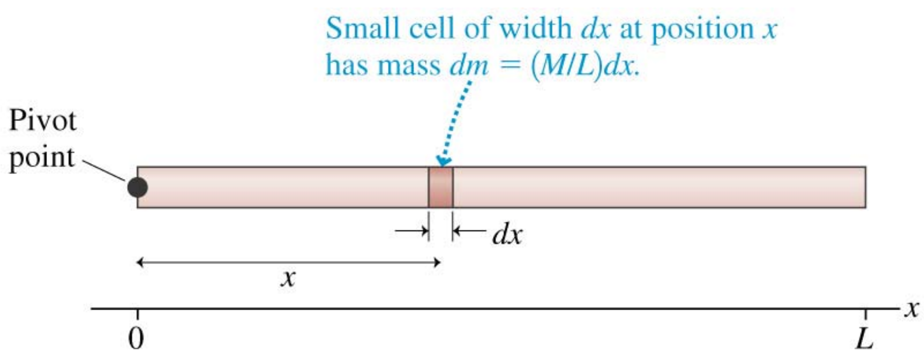
- Then the moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\, dx$$

$$I = \frac{1}{12} ML^2$$

Example 3, cont

Uniform rigid rod
about its end



- The shaded area has a mass

$$dm = \lambda\, dx = (M/L)\, dx$$

- Then the moment of inertia is

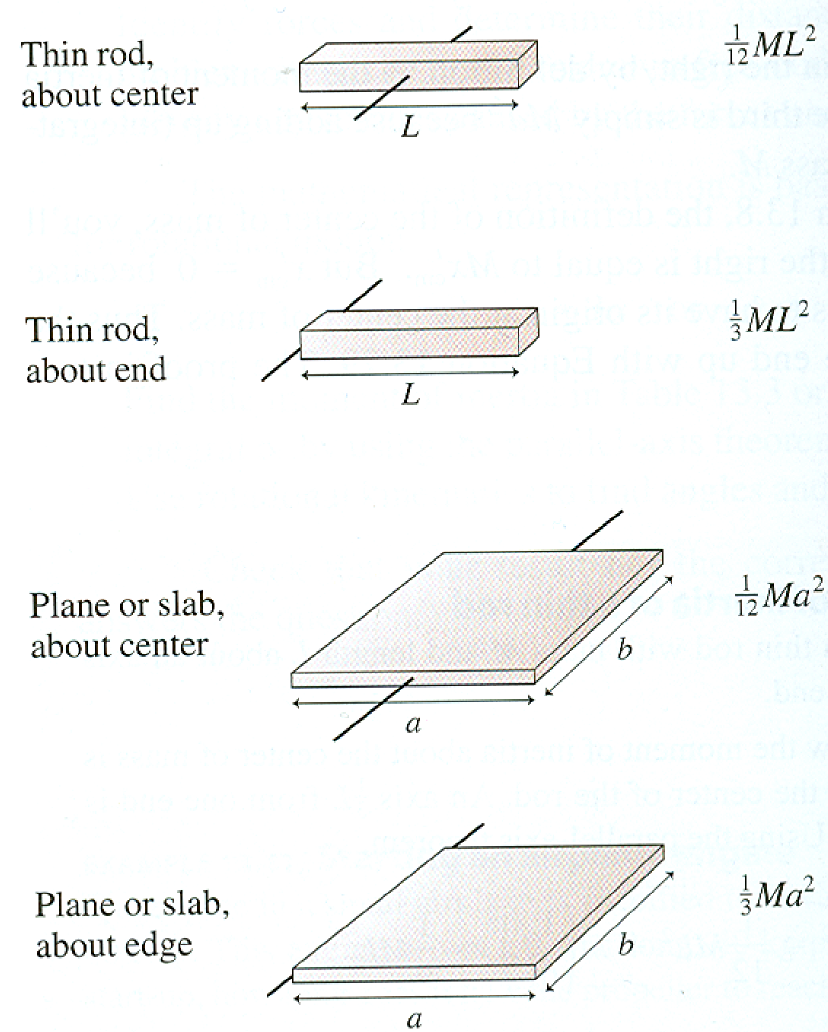
$$I = \int_0^L r^2 dm = \int_0^L x^2 \lambda dx = \lambda \frac{L^3}{3}$$

$$I = \frac{1}{3} M L^2$$

Example 4: *I* of a Slab

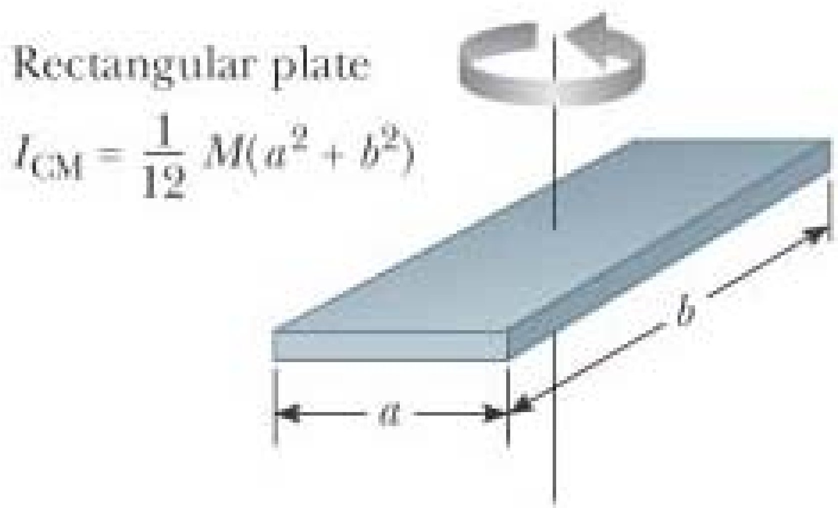
Slab about its center or edge

- The derivation of *I* for the rod did not involve the width or thickness
- Therefore, *the same I* holds for a *rod* or *slab* of the same length and mass



Example 4, cont

Slab about its center



- The mass element is

$$dm = \sigma dx dy = \frac{M}{ab} dx dy$$

- The moment of inertia is

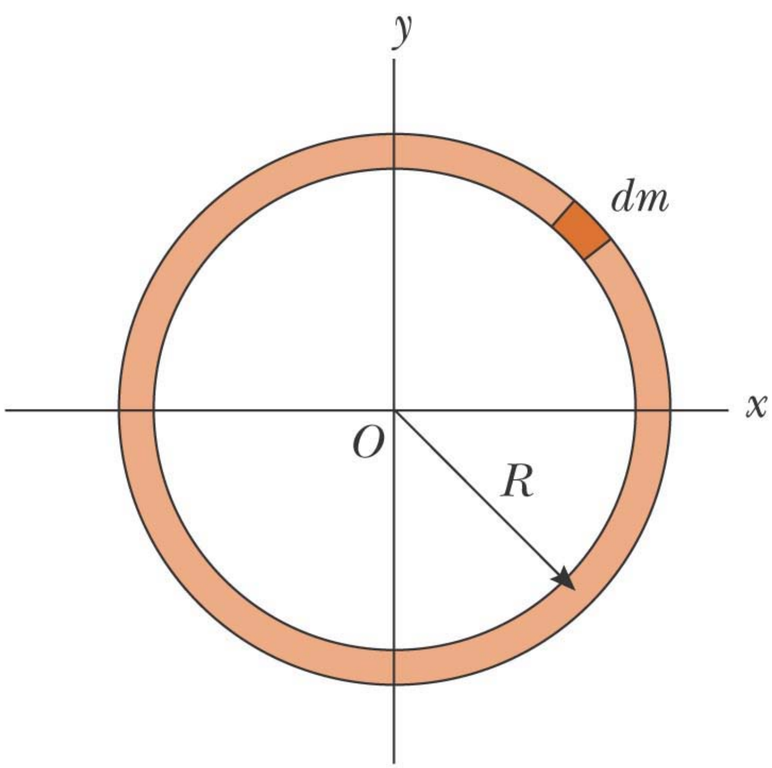
$$I = \int r^2 dm \qquad r^2 = x^2 + y^2$$

$$I = \int_{-a/2}^{+a/2} dx \int_{-b/2}^{+b/2} dy \, \sigma \left(x^2 + y^2\right)$$

$$= \frac{1}{12} M (a^2 + b^2)$$

Example 5: I of a Hoop

Uniform thin hoop
about symmetry axis



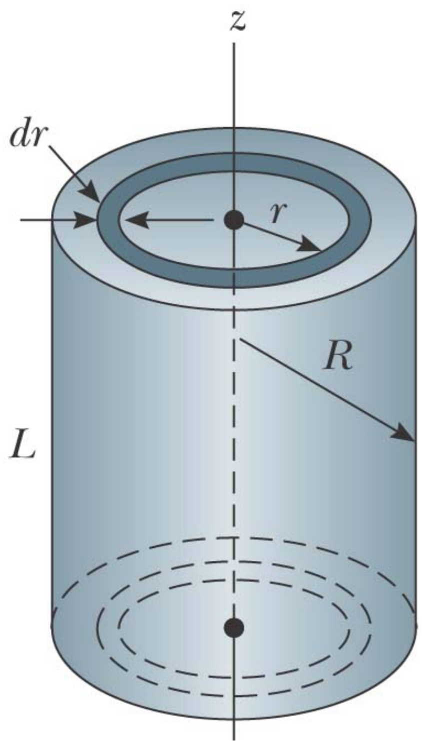
- Since this is a thin hoop, all mass elements are to a good approximation the same distance from the center
- Then the moment of inertia is

$$I = \int r^2 dm = R^2 \int dm$$

$$I = MR^2$$

Example 6: *I* of a Cylinder

Uniform solid cylinder about symmetry axis



- The moment of inertia is

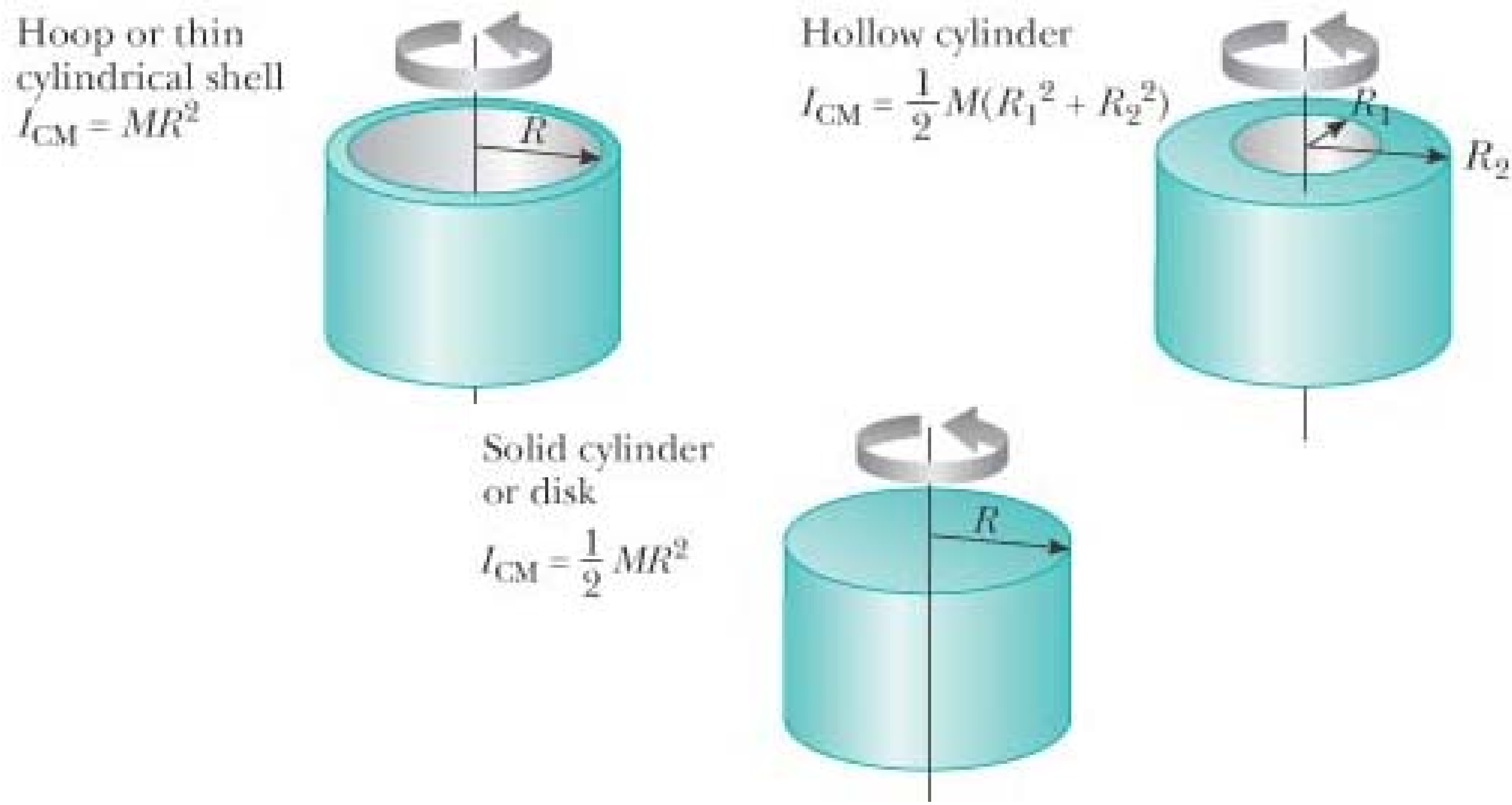
$$I = \int_0^R r^2 dm \qquad dm = \rho dV = \rho \cdot 2\pi r L dr$$

$$I = \int_0^R r^2 \rho 2\pi r L dr = 2\pi L \rho \int_0^R r^3 dr = 2\pi L \rho \frac{R^4}{4}$$

$$= \frac{1}{2}\left(\rho \pi R^2 L\right) R^2 = \frac{1}{2} M R^2$$

Example 6, cont

Cylindrical objects

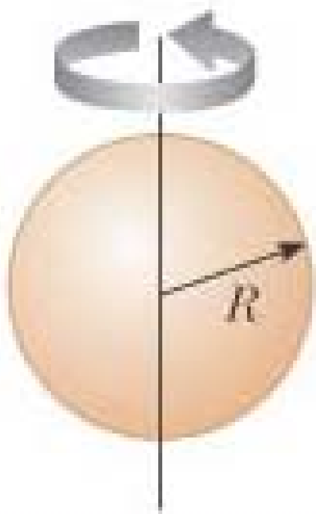


Example 7: I of a Sphere

Spherical objects

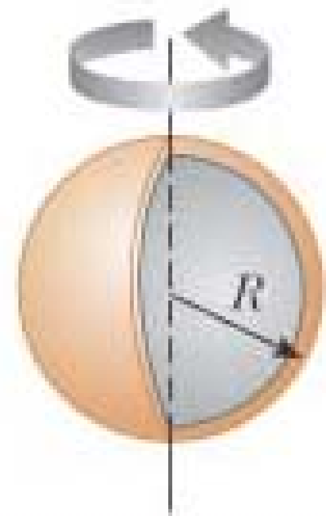
Solid sphere

$$I_{CM} = \frac{2}{5} MR^2$$



Thin spherical shell

$$I_{CM} = \frac{2}{3} MR^2$$

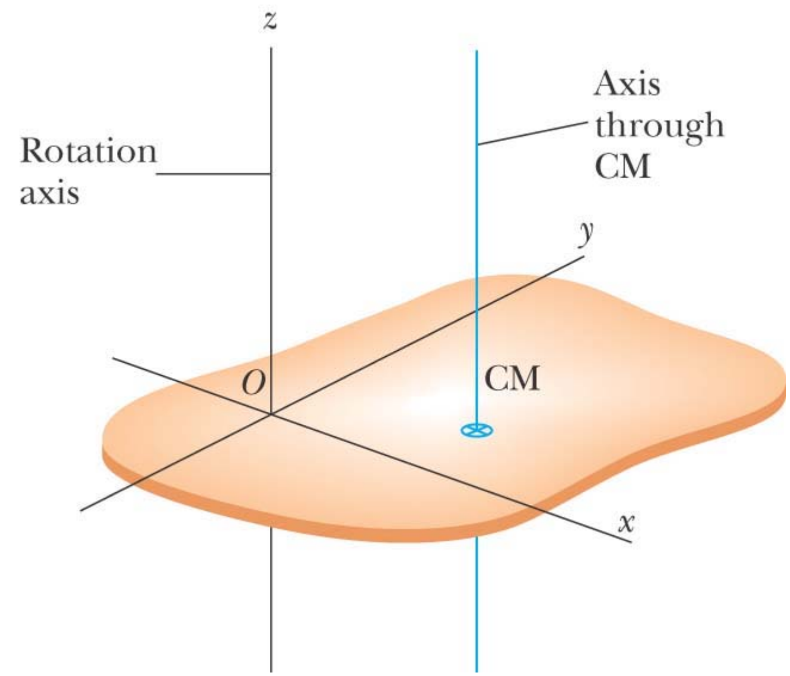


Parallel-Axis Theorem

- In the previous examples, the axis of rotation coincided with the axis of symmetry of the object
- For an *arbitrary* axis, the *parallel-axis theorem* often simplifies calculations
- The theorem states

$$I = I_{CM} + Md^2$$

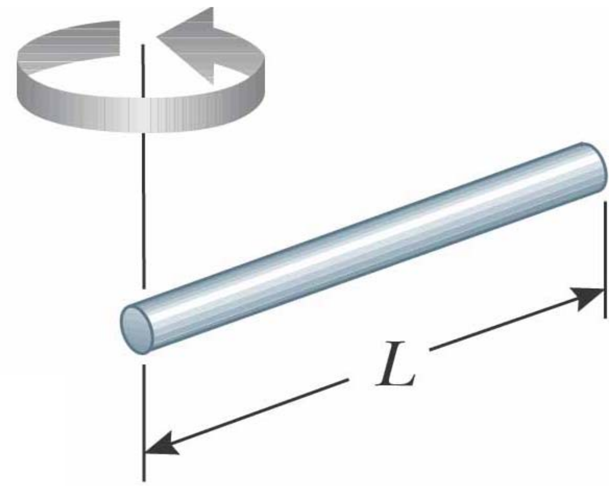
where d is the distance from the CM axis to the rotation axis



Parallel-Axis Theorem, cont

- The moment of inertia of the rod about its center is

$$I_{CM} = \frac{1}{12}ML^2$$



- *D* is 1/2 *L*

$$I = I_{\text{CM}} + MD^2$$

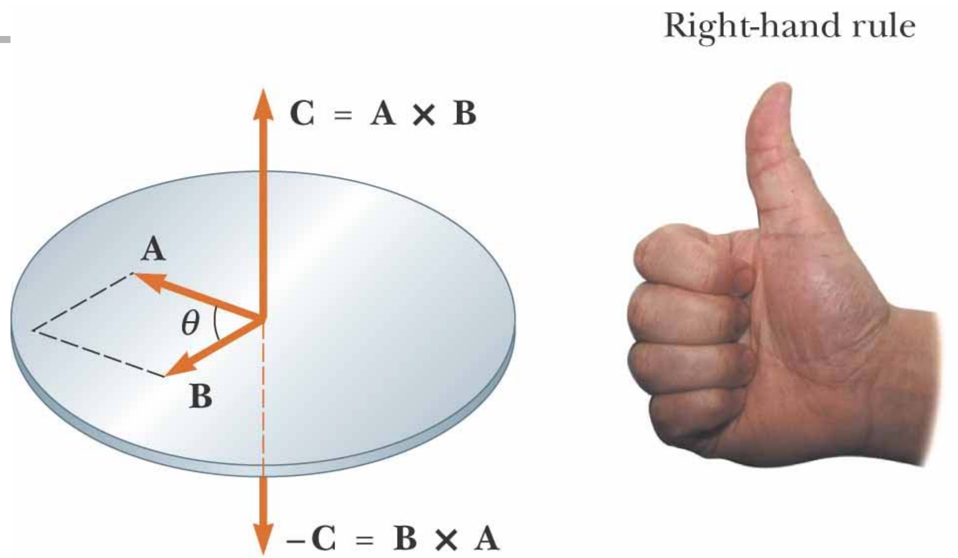
$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

Vector Products of Vectors

- The *vector product* of vectors **A** and **B**, **A** x **B**, is defined as a vector with the magnitude of

  $|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$

  and the direction given by *the right-hand rule*



  - If **A** is parallel to **B**, then **A** x **B** = 0 $\Rightarrow$ **A** x **A** = 0
  - If **A** is perpendicular to **B**, then |**A** x **B**| = AB
- The vector product is also called the *cross product*

Properties of Vector Products

- The vector product is *not commutative*:

  $$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$$

- The vector product obeys the *distributive* law:

  $$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

- The derivative of the cross product is

  $$\frac{d}{dt}\left(\mathbf{A} \times \mathbf{B}\right) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}$$

  - It is important to preserve the multiplicative order of **A** and **B**

Using Components

- The cross product can be expressed as

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}\hat{\mathbf{i}}-\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}\hat{\mathbf{j}}+\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}\hat{\mathbf{k}}$$

  - Expanding the determinants gives

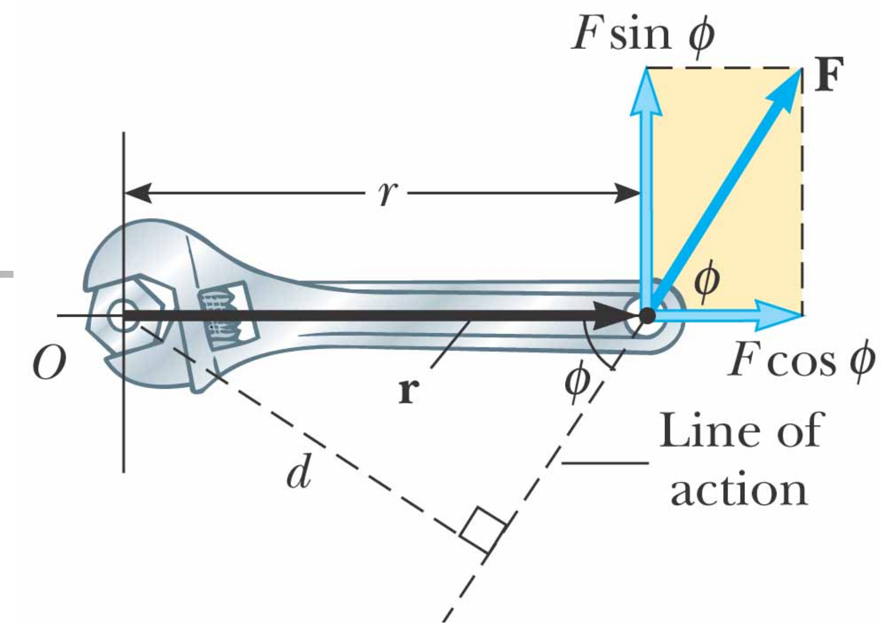
$$\mathbf{A}\times\mathbf{B}=\left(A_yB_z-A_zB_y\right)\hat{\mathbf{i}}-\left(A_xB_z-A_zB_x\right)\hat{\mathbf{j}}+\left(A_xB_y-A_yB_x\right)\hat{\mathbf{k}}$$

- The cross products of the unit vectors:

$$\hat{\mathbf{i}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}\times\hat{\mathbf{k}}=\mathbf{0},$$

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}},\quad \hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}},\quad \hat{\mathbf{k}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}$$

Torque



- A torque is defined as

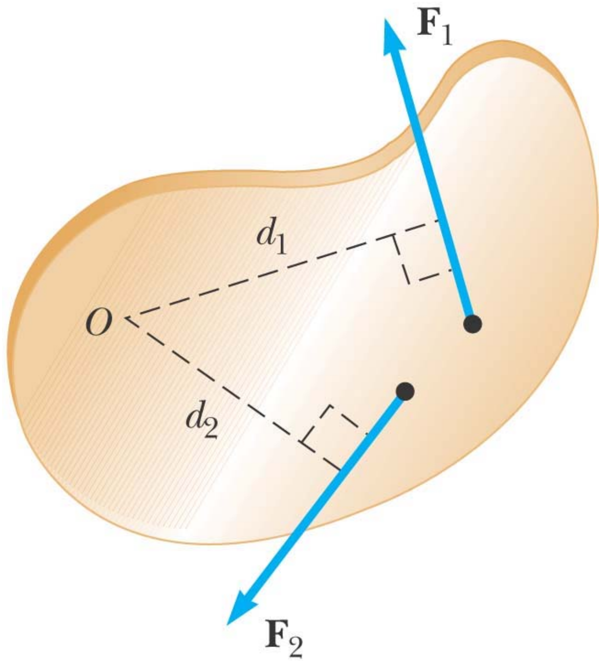
$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

  - $\tau = rF\sin\phi = Fd$
  - The direction of $\boldsymbol{\tau}$ is given by the right hand rule
  - $d = r\sin\phi$ is the *moment arm* (or lever arm), the *perpendicular distance* from the axis of rotation to a line drawn along the direction of the force

- A torque is the tendency of a force to rotate an object about some axis

Net Torque

- Force $\mathbf{F}_1$ will tend to cause a *counterclockwise* rotation about O
  - *Positive* torque
- Force $\mathbf{F}_2$ will tend to cause a *clockwise* rotation about O
  - *Negative* torque
- $\Sigma\tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$



Torque vs Force

- Forces can cause a change in *linear* motion
  - Described by Newton's 2nd Law
- Forces can also cause a change in *rotational* motion
  - The effectiveness of this change depends on the force *and* the moment arm, thus on the *torque*
- The SI units of torque are N·m
  - Although torque is a force multiplied by a distance, it is *very different* from work and energy
  - The units for torque are reported in N·m and *not* changed to Joules

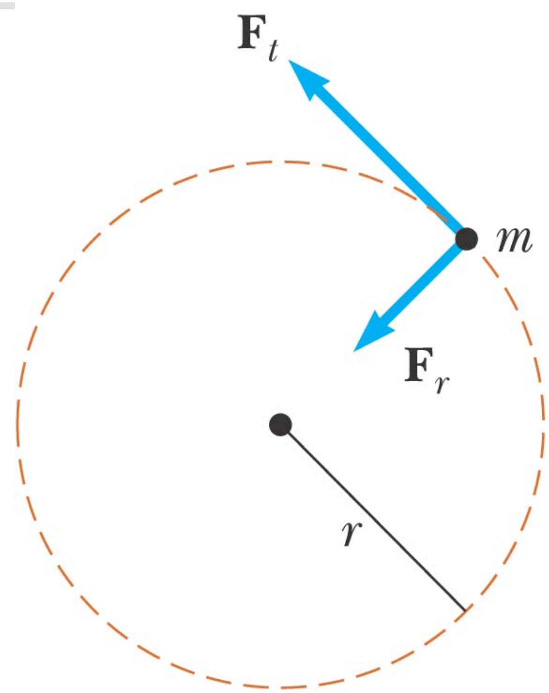
Torque & Angular Acceleration

- The tangential force provides a tangential acceleration:

$$F_t = ma_t = mr\alpha$$

- The torque produced by $\mathbf{F}_t$ about the center is

$$\tau = F_t r = (mr\alpha)r = (mr^2)\alpha$$



- Since mr^2 is the moment of inertia of the particle,

$$\tau = I\alpha$$

  - This is analogous to $F = ma$.

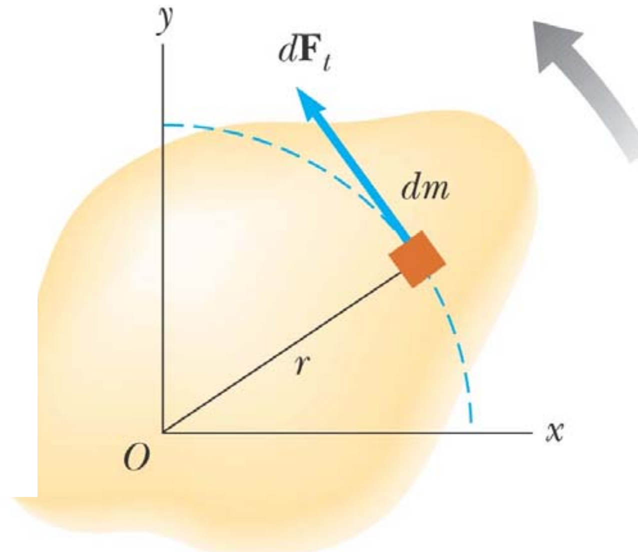
Torque & Angular Acc, Extended

- From Newton's 2nd Law,

$$dF_t = (dm)a_t$$

- The torque associated with the force is

$$d\tau = r\,dF_t = a_t r\,dm = \alpha r^2 dm$$



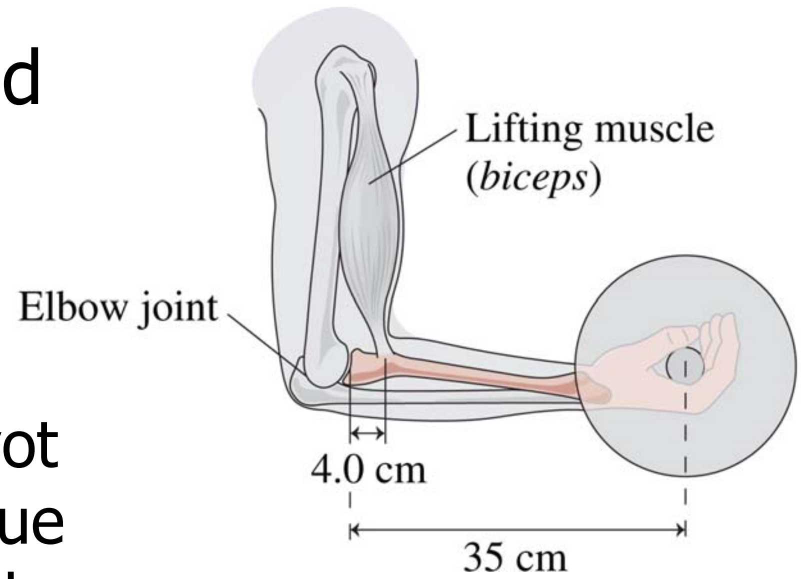
- The net torque is given by

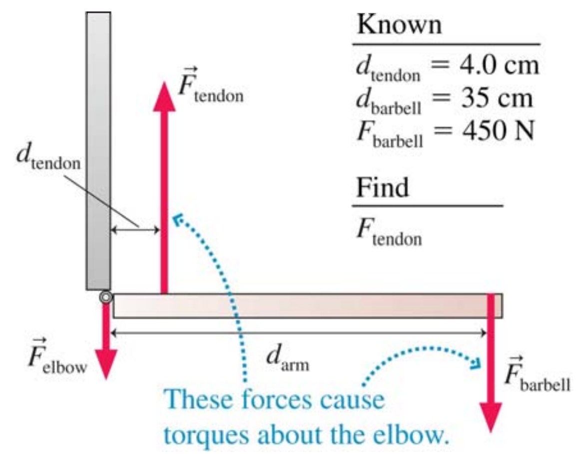
$$\sum \tau = \alpha \int r^2 dm = I\alpha$$

  - This is the same relationship that applied to a particle
  - The result also applies when the forces have radial components

Static Equilibrium

- The condition for a rigid body to be in *static equilibrium* is $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma \tau = 0$
  - You can choose *any* pivot point since the net torque about any point should be zero

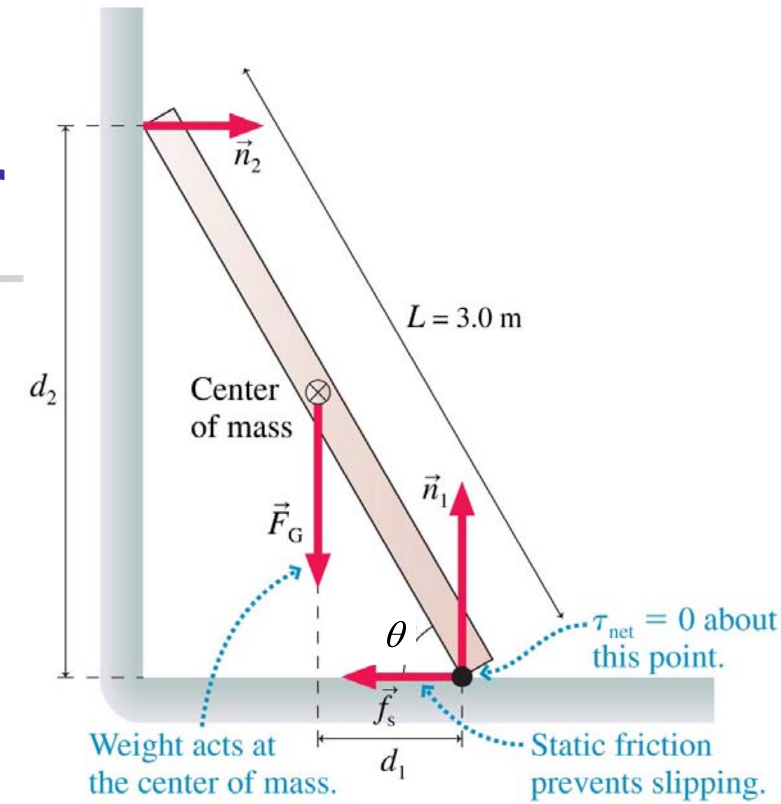




Example 8: Ladder

A 3.0-m-long ladder leans against a frictionless wall at angle θ. The coefficient of static friction with the ground μ_s is 0.20.

What is the minimum angle θ_{min} for which the ladder does not slip?



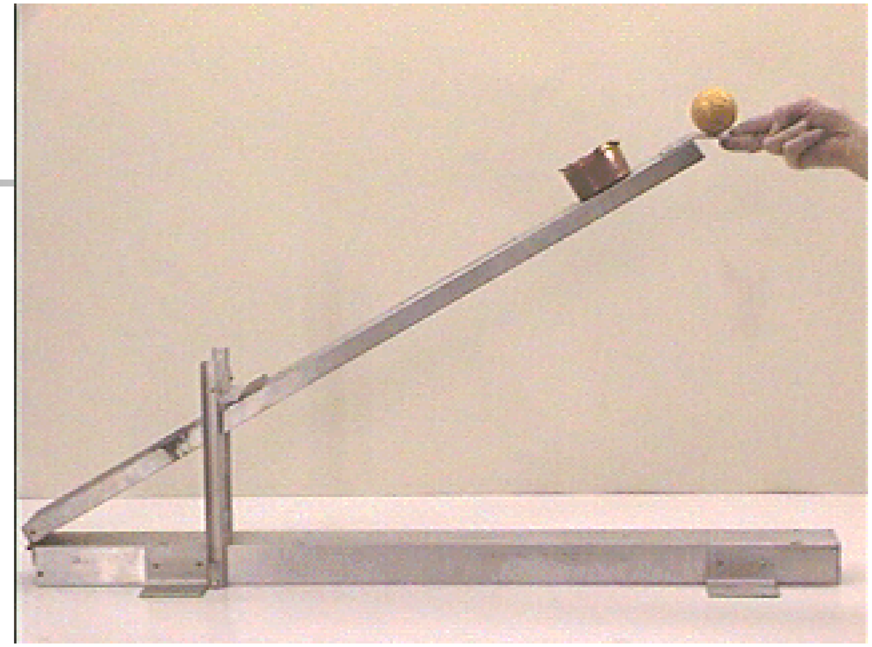
Choose the bottom corner of the ladder as the pivot.

$$\sum F_x = n_2 - f_s = n_2 - \mu_s n_1 = 0, \quad \sum F_y = n_1 - Mg = 0$$

$$\sum \tau = \frac{1}{2}(L\cos\theta)Mg - (L\sin\theta)n_2 = 0$$

$$\tan\theta \geq \frac{Mg}{2n_2} = \frac{Mg}{2\mu_s Mg} = \frac{1}{2\mu_s} = \frac{1}{0.40} = 2.5 \Rightarrow \theta_{min} = \tan^{-1} 2.5 = 68°$$

Hinged Rod



Why does the ball not move with the end of the rod?

For the rod,

$$I = (1/3)mL^2, \; \tau = -mg\,(L/2)\cos\theta$$

$$\tau = I\alpha$$

$$-mg\,(L/2)\cos\theta = 1/3\; mL^2\alpha$$

$$\alpha = -(3/2)(g/L)\cos\theta$$

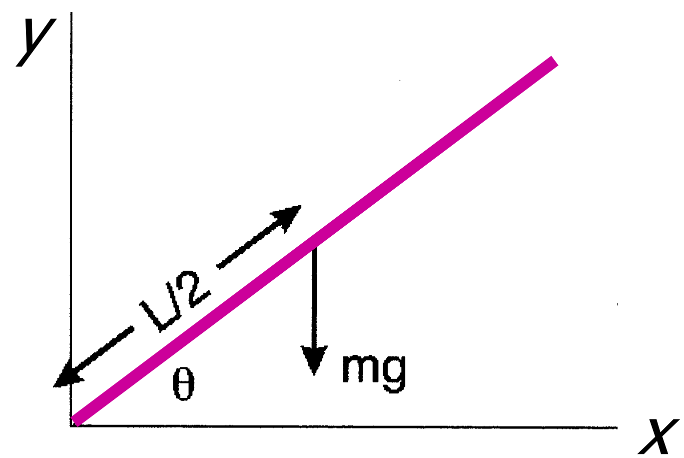
$$a_t = L\alpha = -(3/2)g\cos\theta$$

$$a_y = a_t\cos\theta = -(3/2)g\cos^2\theta$$

For the cup to fall faster than the ball,

$$|a_y| \geq g$$

$$\Rightarrow \; \cos^2\theta \geq 2/3 \quad \theta \leq 35.3°$$



Example 9: Wheel

A block of mass *m* is suspended from a cable which is wrapped around a frictionless wheel of mass *M* and radius *R*.

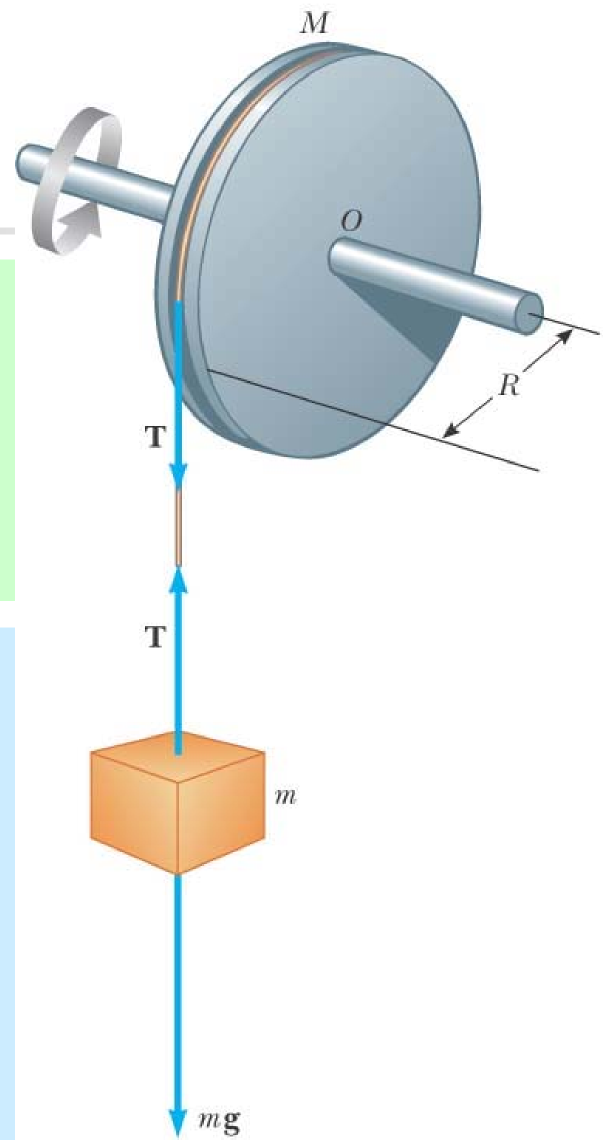
What is the acceleration of the block?

- The wheel is rotating and so we apply

$$\Sigma\tau = I\alpha$$

  - The tension supplies the tangential force
  - Friction between the wheel and string causes the wheel to rotate
- The mass is moving in a straight line, so apply Newton's 2nd Law

$$\Sigma F_y = ma_y = -mg + T$$



Example 9, cont

Wheel: $\Sigma\tau = I\alpha$

$+TR\sin 90° = Ia/R$

$T = Ia/R^2,\ I = (1/2)MR^2$

Mass: $T - mg = -ma$

Solve for a:

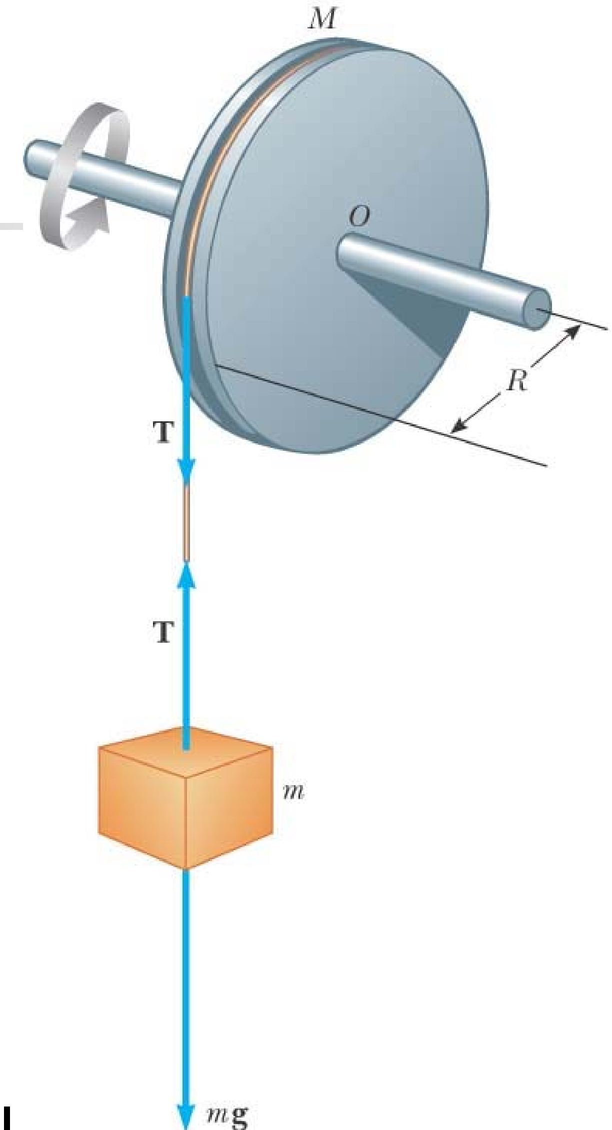
$a = -T/m + g = -Ia/mR^2 + g$

$a = g/(1 + I/mR^2)$

$a < g$ for $M > 0$

$T = mg/(1 + mR^2/I) = mg/(1 + 2m/M)$

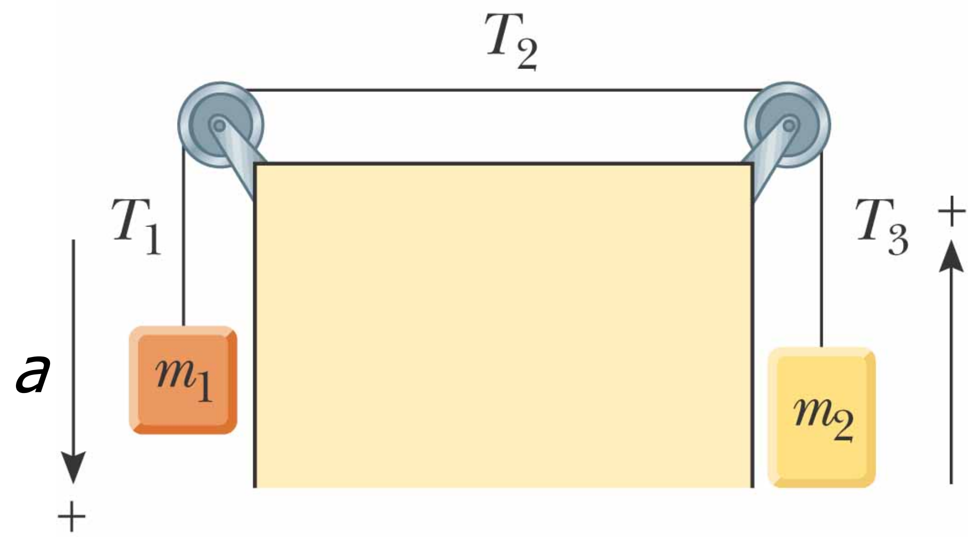
As $M \to 0$, $T \to 0$, i.e. m is in free fall



Example 10: Pulleys and Masses

Two masses, m_1 and m_2, are connected to each other through two identical massive pulleys, as shown in the figure.

Find the acceleration of the masses.



- Both masses move in linear directions, so apply Newton's 2nd Law
- Both pulleys rotate, so apply the torque equation
- Combine the results

Example 10, cont

Newton's 2nd Law for the masses

$$T_1 - m_1 g = -m_1 a, \quad T_3 - m_2 g = m_2 a$$

$$\Rightarrow \; T_3 - T_1 = (m_1 - m_2)g - (m_1 + m_2)a$$

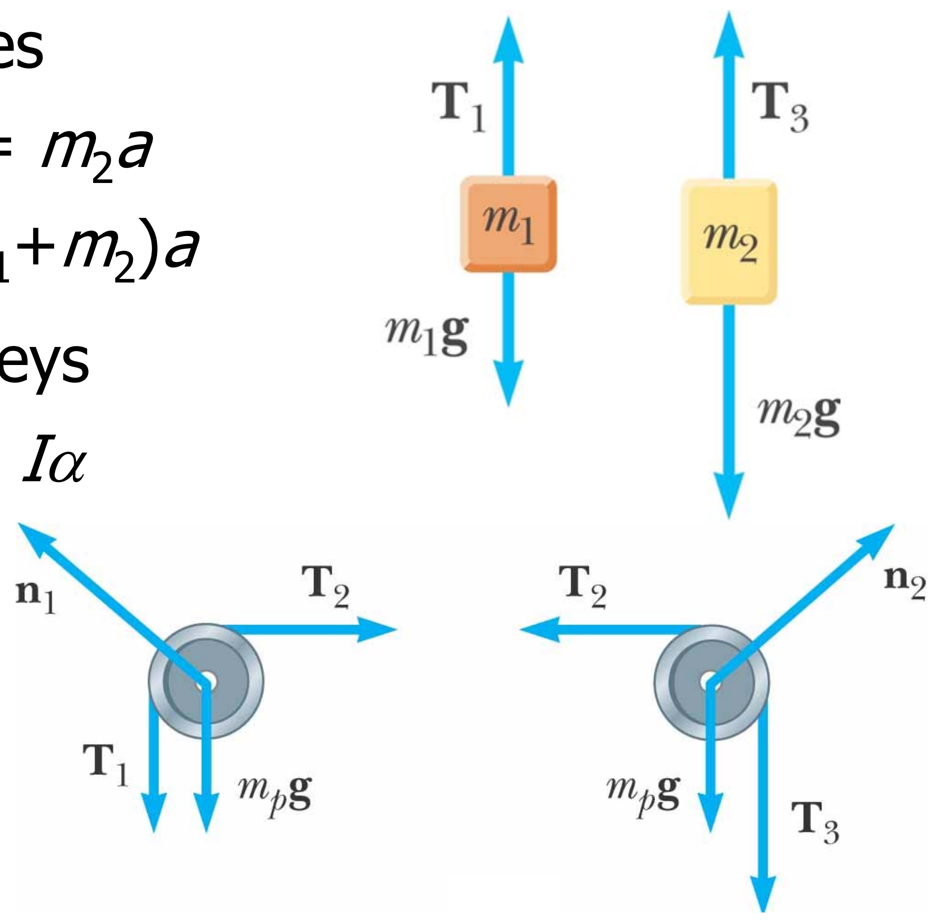
The torque equation for the pulleys

$$(T_1 - T_2)R = I\alpha, \; (T_2 - T_3)R = I\alpha$$

$$\Rightarrow \; T_1 - T_3 = 2I\alpha = 2Ia/R$$

Combine these results to obtain

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + 2(I/R^2)}$$



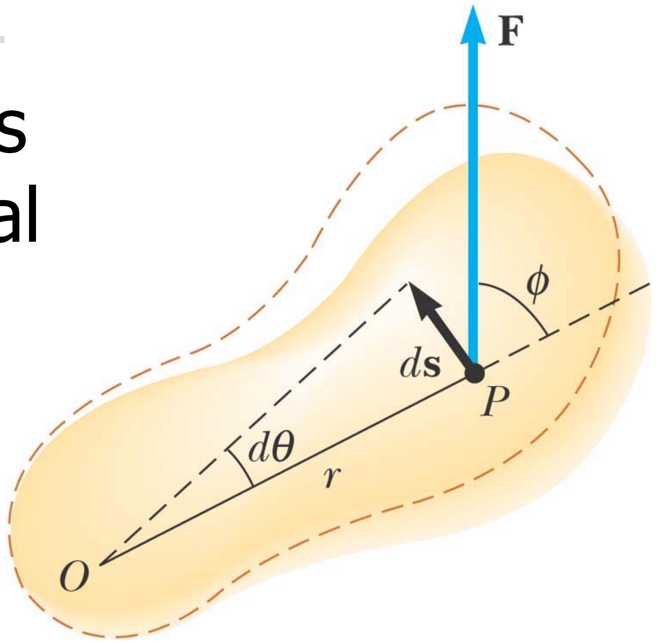
Work in Rotational Motion

- Work done by **F** on the object as it rotates through an infinitesimal distance $ds = rd\theta$ is

$$dW = \mathbf{F} \cdot d\,\mathbf{s} = (F \sin\phi)\ rd\theta$$

$$\Rightarrow \quad dW = \tau d\theta$$

  - The radial component of **F** does no work because it is perpendicular to the displacement



Power in Rotational Motion

- The rate at which work is being done in a time interval *dt* is

$$\text{Power} = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$

- This is analogous to $P = \mathbf{F}\cdot\mathbf{v}$ in a linear system

Work-Kinetic Energy Theorem

- The net work done by external forces in rotating a rigid object *about a fixed axis* equals the change in the object's *rotational* kinetic energy:

$$\sum W = \int \tau d\theta = \int I\alpha d\theta = \int I \frac{d\omega}{dt}\omega dt = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2$$

- In general, net work done by external forces on an object is the change in its *total* kinetic energy, the sum of translational and rotational kinetic energies:

$$\sum W = \Delta K_{tot} = \left(\frac{1}{2} m v_f^2 + \frac{1}{2} I\omega_f^2\right) - \left(\frac{1}{2} m v_i^2 + \frac{1}{2} I\omega_i^2\right)$$

Energy Conservation

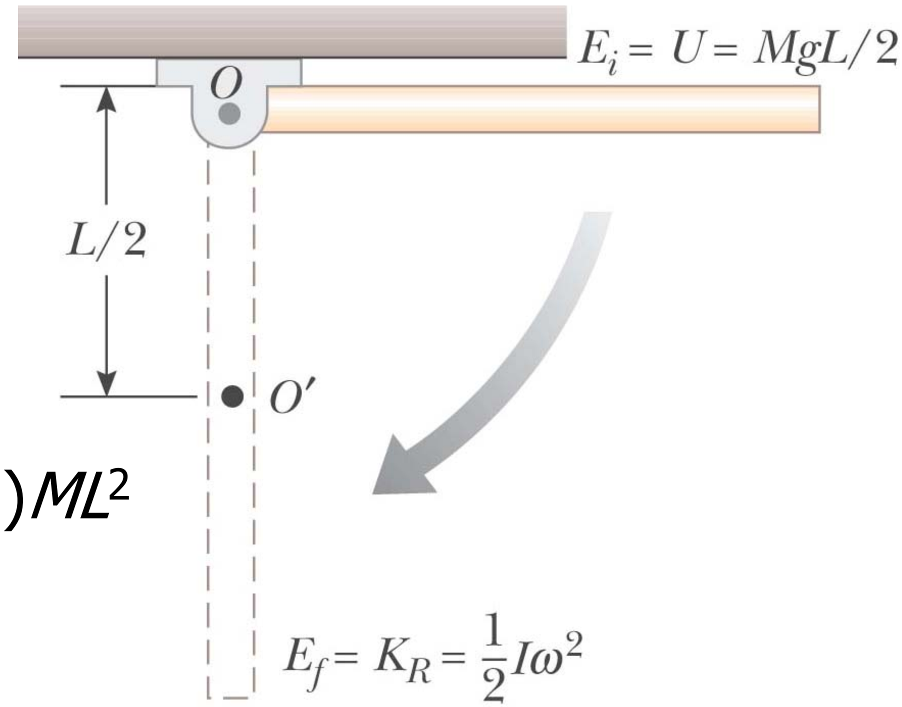
- If *no friction* is involved, *total mechanical energy* is conserved

$$K_f + U_f = K_i + U_i$$

  - K includes both rotational and translational kinetic energies, if appropriate

Example 11: Rotating Rod

What is the rod's angular speed when it reaches its lowest position?



Two methods:

1) $\sum W = K_f - K_i$

$MgL/2 = (1/2)I\omega_f^2$, $I = (1/3)ML^2$

$\Rightarrow \quad \omega_f = (3g/L)^{1/2}$

2) Since there is no friction mentioned, assume that energy is conserved.

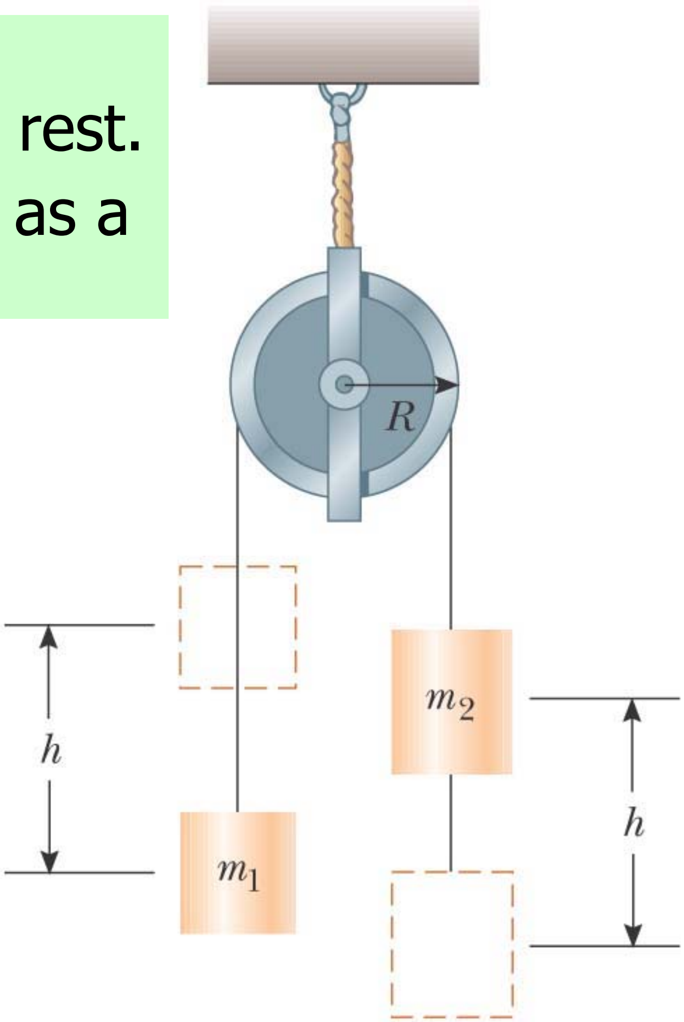
$K_f + U_f = K_i + U_i$

$(1/2)I\omega_f^2 + 0 = 0 + MgL/2 \Rightarrow \omega_f = (3g/L)^{1/2}$

Example 12: Atwood Machine

Two masses, m_1 and m_2, connected by a string around a pulley, are released from rest.

Find the linear and the angular velocities as a function of h.



Assume energy is conserved:

$$\Delta E = \Delta K + \Delta U = 0$$

$$K_i = 0,\ K_f = \tfrac{1}{2}(m_1 + m_2)v^2 + \tfrac{1}{2} I\omega^2$$

$$\Delta U = U_f - U_i = -m_1 gh + m_2 gh$$

$$\Rightarrow \tfrac{1}{2}(m_1 + m_2)v^2 + \tfrac{1}{2} I\omega^2 + (m_2 - m_1)gh = 0$$

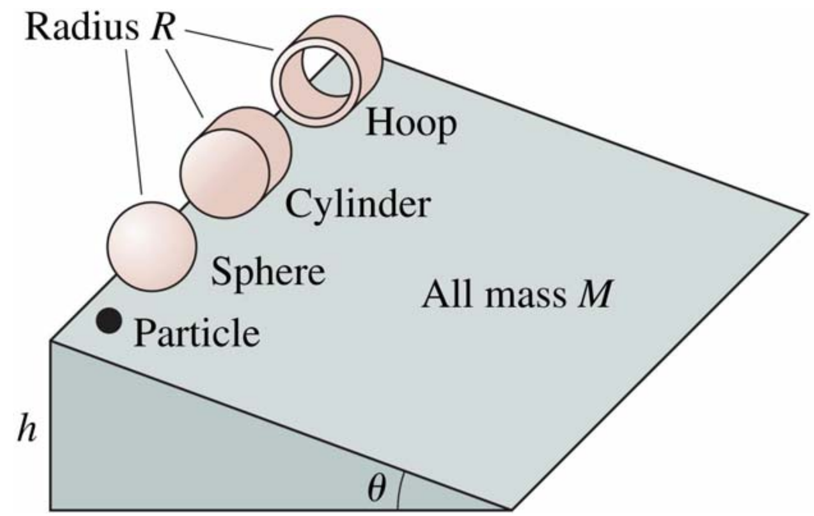
Substituting $\omega = v/R$,

$$v = \sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2 + (I/R^2)}} \cong \sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2 + M}}$$

Example 13: Downhill Race

A sphere, a cylinder, and a circular hoop, all of mass M and radius R, are placed at height h on a slope of angle θ.

Which one will win the race to the bottom of the hill?



For work done by gravitational force: $W_g = mgh$

$$W = K_f - K_i = (½I\omega_f^2 + ½mv_f^2) - (½I\omega_i^2 + ½mv_i^2)$$

If no slipping occurs, $\omega = v/R$. Also, $\omega_i = 0$.

$$W = ½\,(I/R^2 + m)v_f^2,\ v_f = (2W_g/(I/R^2 + m))^{1/2} = 2gh/(I/mR^2+1)^{1/2}$$

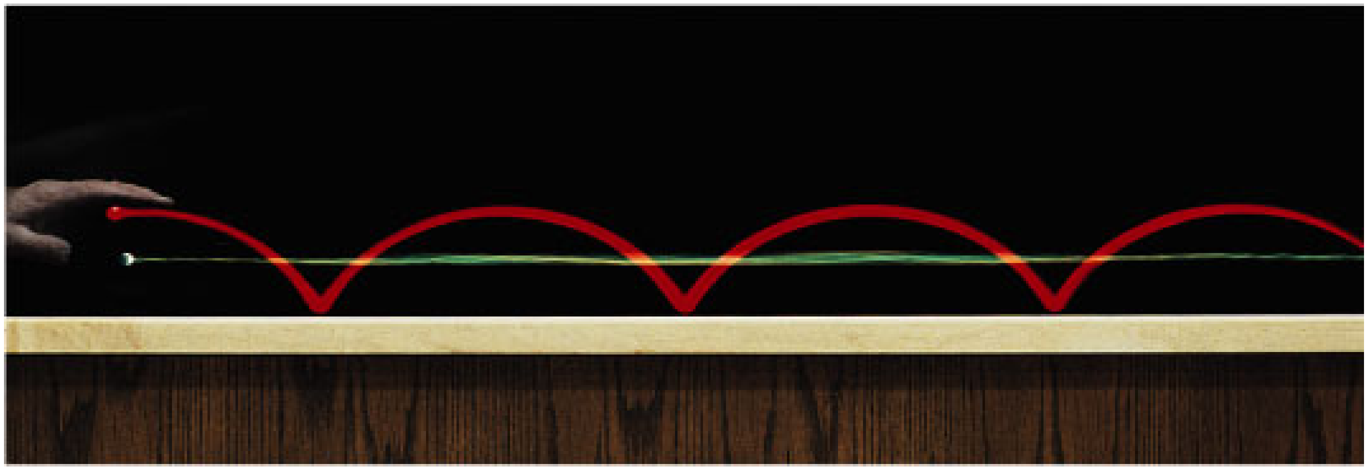
The smaller I, the bigger v_f.

I/mR^2 values: Particle: 0, Sphere: 2/5, Cylinder: ½, Hoop: 1

Summary of Useful Equations

Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Net torque $\Sigma\tau = I\alpha$	Net force $\Sigma F = ma$
If α = constant $\begin{cases} \omega_f = \omega_i + \alpha_t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If a = constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$	Work $W = \int_{x_i}^{x_f} F_x\, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

Rolling Cylindrical Object



- The red curve shows the path moved by a point on the rim of the object
  - This path is called a *cycloid*
- The green line shows the path of the CM of the object

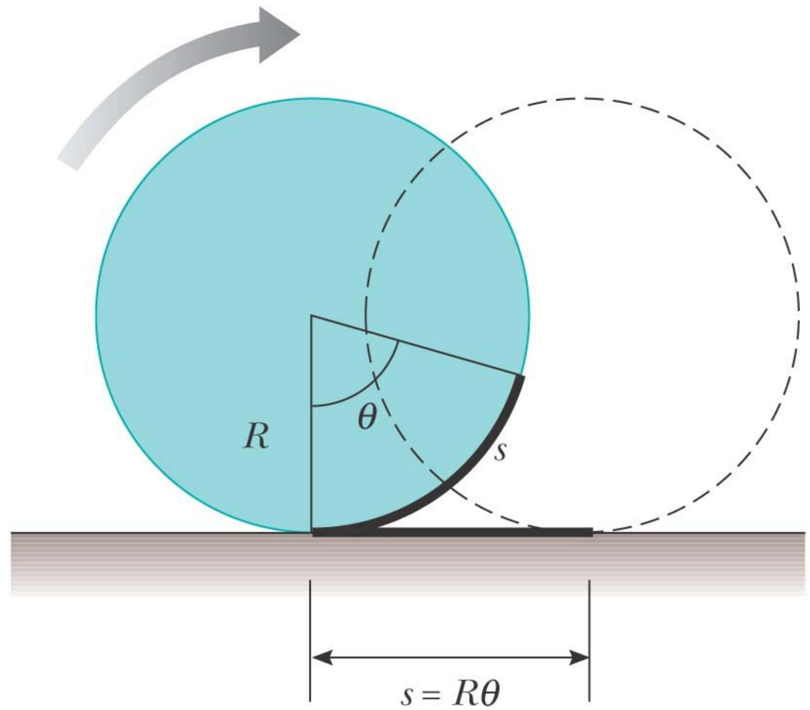
Rolling Object, No Slipping

- The velocity of the CM is

$$v_{\mathrm{CM}} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

- The acceleration of the CM is

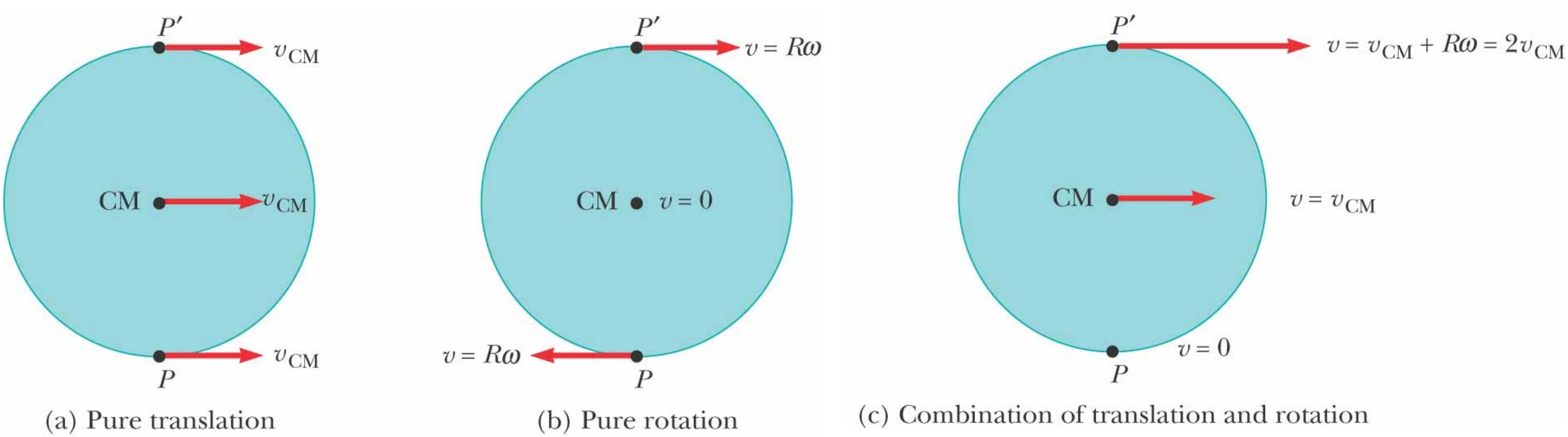
$$a_{\mathrm{CM}} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



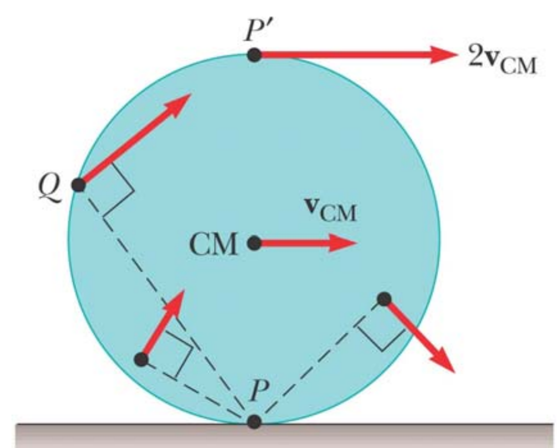
- Rolling motion can be modeled as a *combination* of pure *translational* motion & pure *rotational* motion

Rolling Motion

- Relationship between rotation and translation



- At any instant, the point on the rim located at point *P* is *at rest* relative to the surface since no slipping occurs



Kinetic Energy of a Rolling Object

- The total kinetic energy of a rolling object is the *sum* of the translational energy of its CM and the rotational energy about its CM

$$K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2 = \frac{1}{2} I_{CM} \left(\frac{v_{CM}}{R} \right)^2 + \frac{1}{2} M v_{CM}^2$$

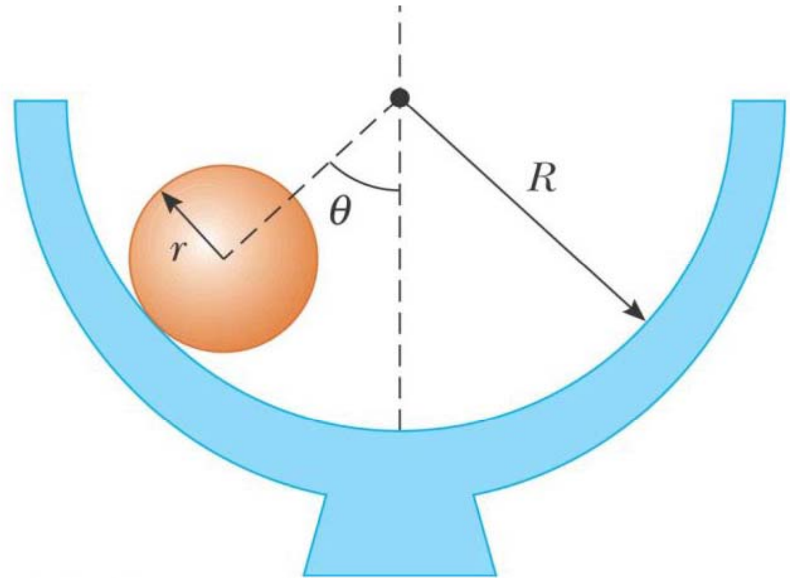
$$= \frac{1}{2} M v_{CM}^2 \left(\frac{I_{CM}}{MR^2} + 1 \right)$$

- Accelerated rolling motion is possible only if rolling friction is present
  - The friction produces the torque required for rotation

Example 14: Ball in a Sphere

A uniform solid sphere of radius r is placed on the inside surface of a hemispherical bowl with much larger radius R. The sphere is released from rest at an angle θ to the vertical and rolls without slipping.

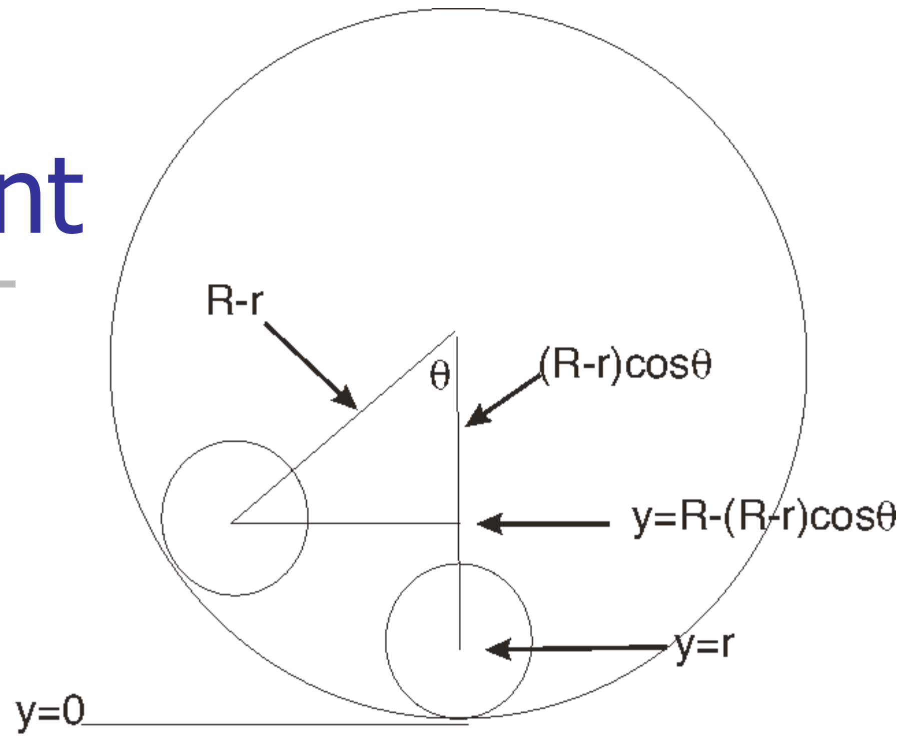
Determine the speed of the sphere when it reaches the bottom of the bowl.



Mechanical energy is conserved

$I = (2/5)mr^2$ for a solid sphere

Example 14, cont



Energy is conserved:

$$U_f + K_f = U_i + K_i$$

$$U_i = mg[R - (R - r)\cos\theta]$$

$$U_f = mgr$$

$$K_i = 0$$

$$K_f = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}mv^2(1 + I/mr^2)$$

$$= \tfrac{1}{2}mv^2(1 + 2/5) = (7/10)mv^2$$

Substituting *U*'s and *K*'s into energy conservation,

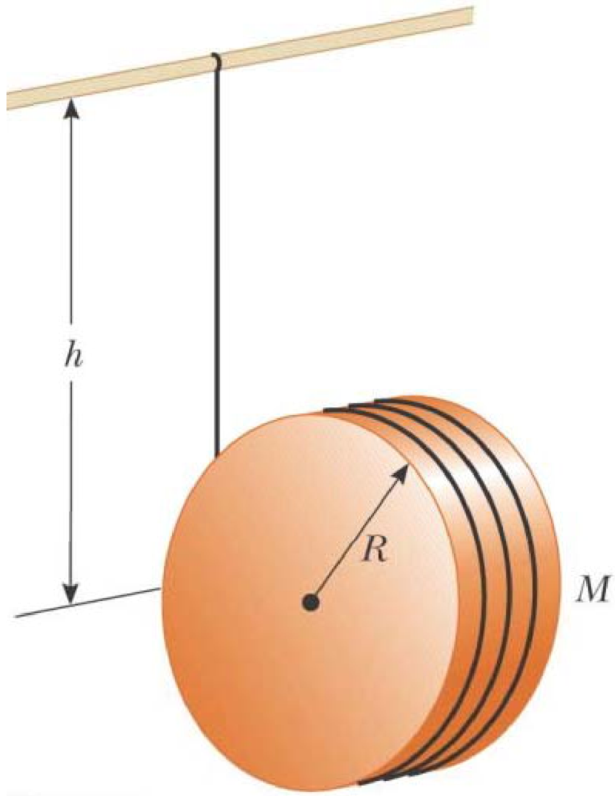
$$mg[R - (R - r)\cos\theta] - mgr = (7/10)mv^2$$

$$\Rightarrow v = [(10/7)g(R - r)(1 - \cos\theta)]^{1/2}$$

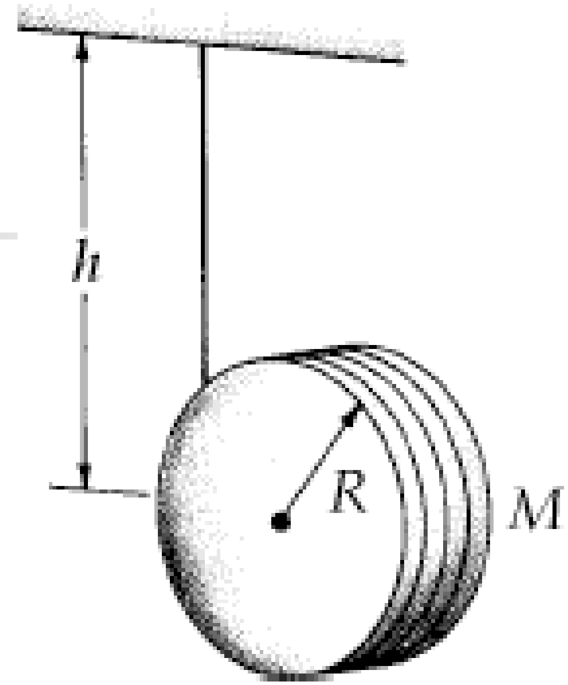
Example 15: Falling Disk

A string is wound around a uniform disk of radius R and mass M. The disk is released from rest with the string vertical and its top end tied to a fixed bar.

Show that (a) the tension in the string is one-third the weight of the disk, (b) the magnitude of the acceleration of the CM is $2g/3$, and (c) the speed of the CM is $(4gh/3)^{1/2}$ after the disk has descended through distance h. (d) Verify your answer to (c) using the energy approach.



Example 15, cont



(a) $\sum F_y = T - Mg = -Ma \quad \Rightarrow \quad T = M(g-a)$

$$\sum \tau = -TR = -I\alpha = -\frac{1}{2}MR^2\left(\frac{a}{R}\right) \quad \Rightarrow \quad a = \frac{2T}{M}$$

Solving these equations for T, $T = \frac{Mg}{3}$

(b) $a = \frac{2T}{M} = \frac{2}{M}\left(\frac{Mg}{3}\right) = \frac{2}{3}g$

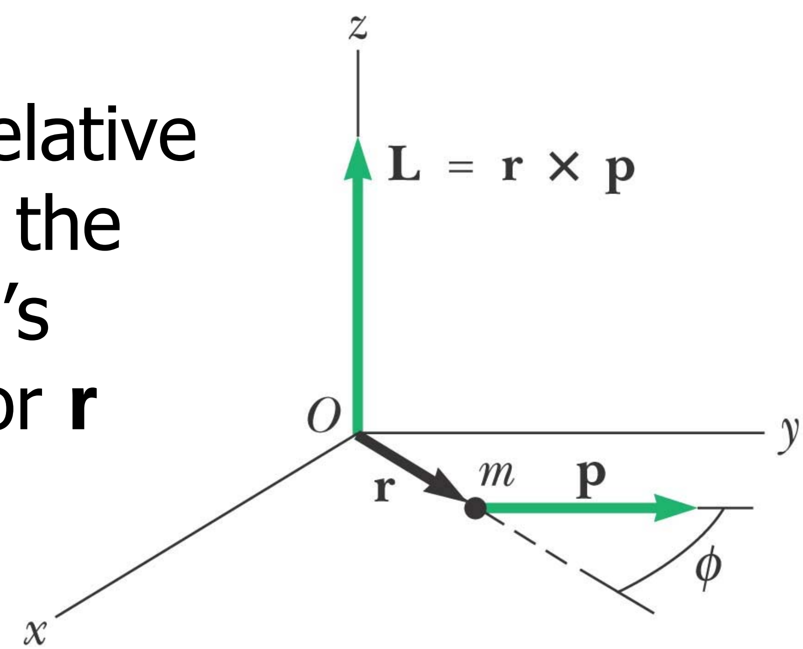
(c) $v_f^2 = v_i^2 + 2a\left(x_f - x_i\right) = 0 + 2\left(\frac{2}{3}g\right)(h-0) \quad \Rightarrow \quad v_f = \sqrt{\frac{4}{3}gh}$

(d) Energy conservation : $U_{gf} + K_{tra,f} + K_{rot,f} = U_{gi} + K_{tra,i} + K_{rot,i}$

$$0 + \frac{1}{2}Mv_f^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_f}{R}\right)^2 = Mgh + 0 + 0 \quad \Rightarrow \quad v_f = \sqrt{\frac{4}{3}gh}$$

Angular Momentum

- The *instantaneous angular momentum* **L** of a particle relative to the origin *O* is defined as the *cross product* of the particle's instantaneous position vector **r** and its instantaneous linear momentum **p**:

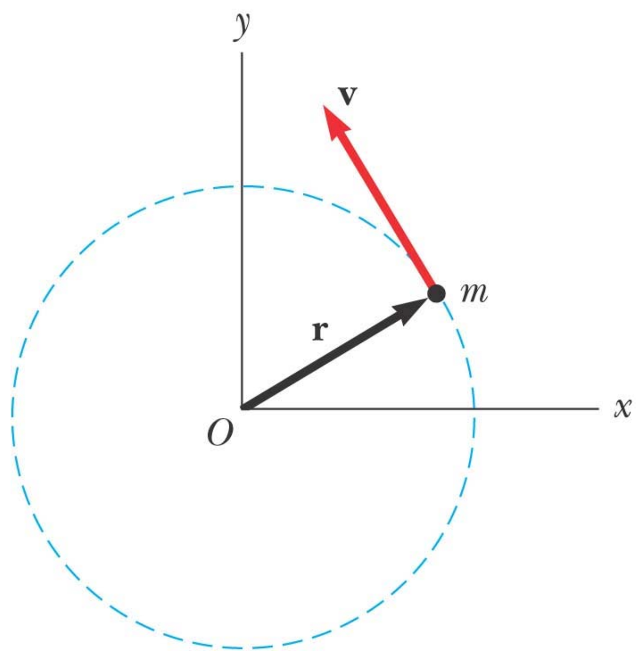


$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

- The SI units of angular momentum are (kg·m^2)/s

Angular Momentum, cont

- Both the magnitude and direction of **L** depend on the choice of origin
  - The magnitude of **L** is $mvr \sin\phi$
  - The direction of **L** is perpendicular to the plane formed by **r** and **p**
- Example: Uniform circular motion
  - **L** is pointed out of the diagram
  - $L = mvr \sin 90^o = mvr$
  - **L** is constant and is along an axis through the center of its path



Torque and Angular Momentum

- Consider a particle of mass *m* located at the vector position **r** and moving with linear momentum **p**

$$\mathbf{r} \times \sum \mathbf{F} = \sum \tau = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

$$\text{Adding the term } \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

$$\sum \tau = \frac{d(\mathbf{r} \times \mathbf{p})}{dt}$$

$$\text{Note } \frac{d\mathbf{r}}{dt} = \mathbf{v}$$

$$\therefore \frac{d\mathbf{r}}{dt} \times \mathbf{p} = \mathbf{v} \times m\mathbf{v} = \mathbf{0}$$

Torque & Ang Momentum, cont

- The torque is related to the angular momentum similar to the way force is related to linear momentum:

$$\sum \tau = \frac{d\mathbf{L}}{dt}$$

- This is another rotational analog of Newton's 2nd Law
  - $\Sigma\tau$ and **L** must be measured about the same origin
  - This is valid for any origin *fixed in an inertial frame*, i.e. not accelerating

L of a System of Particles

- Differentiating $\mathbf{L}_{tot}$ with respect to time,

$$\frac{d\mathbf{L}_{\text{tot}}}{dt} = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \tau_i$$

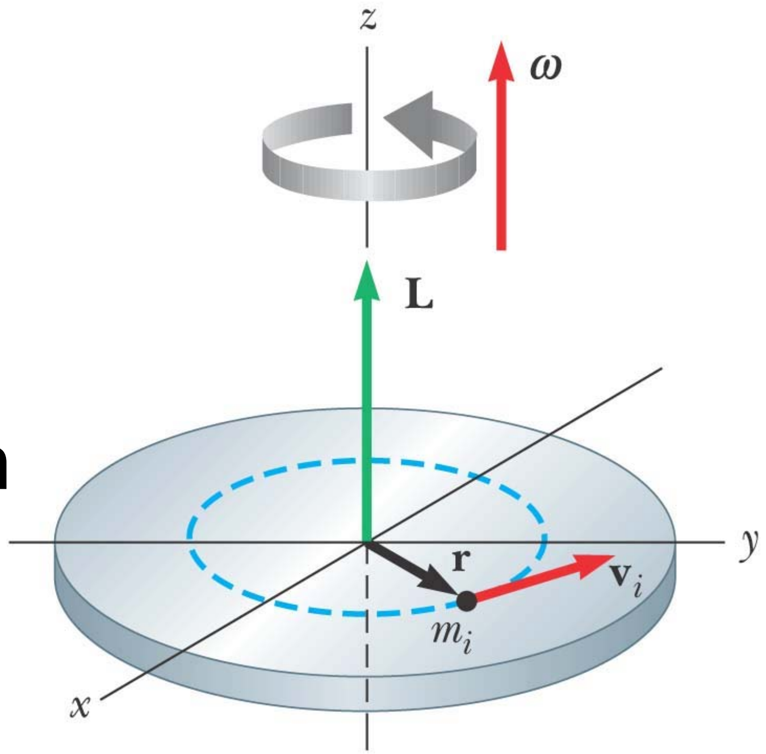
- Any torques associated with the internal forces acting in a system of particles are zero:

$$\sum \tau_{\text{ext}} = \frac{d\mathbf{L}_{\text{tot}}}{dt}$$

- The resultant torque acting on a system about an axis through the CM equals $d\mathbf{L}_{tot}/dt$ of the system *regardless of* the motion of the CM

L of a Rotating Rigid Object

- Each particle of the object rotates in the *xy* plane about the *z* axis with an angular speed of ω
- The angular momentum of an individual particle is:
  $$\left|\mathbf{L}_i\right| = \left|\mathbf{r}_i \times (m_i \mathbf{v}_i)\right| = m_i r_i^2 \omega$$
- **L** and $\boldsymbol{\omega}$ are directed along the *z* axis



L of a Rotating Rigid Object, cont

- To find the angular momentum of the entire object, add the angular momenta of the individual particles

$$L_z = \sum_i L_i = \sum_i m_i r_i^2 \omega = I\omega$$

- This gives the rotational form of Newton's 2nd Law

$$\sum \tau_{\text{ext}} = \frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha$$

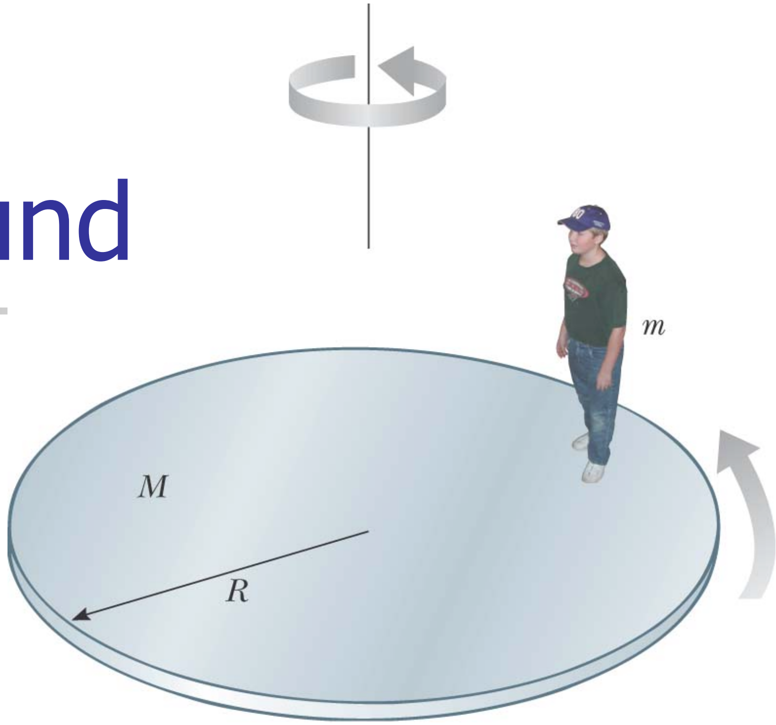
- If a *symmetrical* object rotates about *a symmetry axis*, the vector form holds with I as a scalar:

$$\mathbf{L} = I\boldsymbol{\omega}$$

Conservation of Ang Momentum

- The total angular momentum of a system is constant in both magnitude and direction if *the net external torque* acting on the system is *zero*
  - $\Sigma\tau_{ext} = 0$ means the system is isolated
- If the mass of an isolated system undergoes redistribution, the moment of inertia changes
  - The conservation of angular momentum requires a compensating *change in the angular velocity*
  - $I_i \, \omega_i = I_f \, \omega_f$

The Merry-Go-Round



- $I_{\text{system}} = I_{\text{platform}} + I_{\text{person}}$
- As the person moves toward the center of the rotating platform, the angular speed will increase
  - To keep **L** constant
- As the figure skater retracts her stretched arms while spinning, her angular speed increases

Intrinsic Angular Momentum

- Angular momentum has been used in the development of modern theories of atomic, molecular and nuclear physics
- The angular momentum has been found to be an *intrinsic* property of these objects
- Angular momenta are multiples of a fundamental unit of angular momentum

$$\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34} \frac{\mathrm{kg \cdot m^2}}{\mathrm{s}}$$

  - *h* is called the Planck constant

Example 16: Woman on Turntable

A 60.0-kg woman stands at the rim of a horizontal turntable having a moment of inertia of 500 $kg \cdot m^2$ and a radius of 2.00 m. The turntable is *initially at rest* and is free to rotate about a frictionless, vertical axle through its center. The woman then starts walking around the rim clockwise (as viewed from above the system) at a constant speed of 1.50 m/s relative to the Earth.

(a) In what direction and with what angular speed does the turntable rotate? (b) How much work does the woman do to set herself and the turntable into motion?

Example 16, cont

Initially the turntable is not moving so its angular momentum is zero. Since there is no external torque on the table, the angular momentum stays zero when the woman walks. Since she walks cw, $\omega_w < 0$.

$$\text{(a)}\ L_{woman} + L_{turntable} = 0,\quad I_w\omega_w + I_t\omega_t = 0,\quad I_t\omega_t = -m_w r^2(-\frac{v}{r})$$

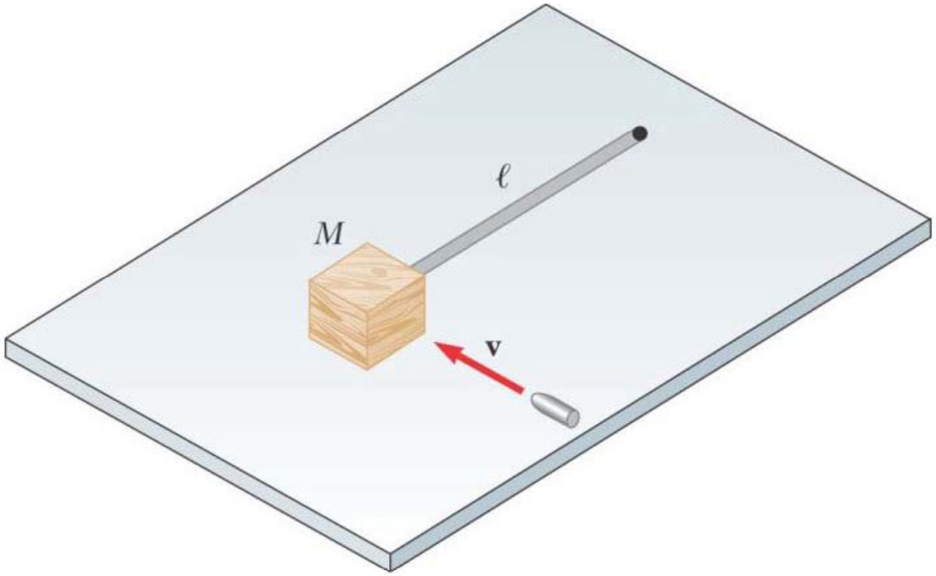
$$\omega_t = +\frac{m_w r v}{I_i} = \frac{60.0\ \text{kg}(2.00\ \text{m})(1.50\ \text{m/s})}{500\ \text{kg m}^2} = 0.360\ \text{rad/s ccw}$$

$$\text{(b)}\ W = \Delta K = K_f - 0 = \frac{1}{2}m_{woman}v_{woman}^2 + \frac{1}{2}I\omega_{table}^2$$

$$W = \frac{1}{2}(60.0\ \text{kg})(1.50\ \text{m/s})^2 + \frac{1}{2}(500\ \text{kg m}^2)(0.360\ \text{rad/s})^2 = 99.9\ \text{J}$$

Example 17: Bullet and Block

A wooden block of mass M resting on a frictionless horizontal surface is attached to a rigid rod of length ℓ and of negligible mass. The rod is pivoted at the other end. A bullet of mass m traveling parallel to the horizontal surface and normal to the rod with speed v hits the block and becomes embedded in it.



(a) What is the angular momentum of the bullet-block system?

(b) What fraction of the original kinetic energy is lost in the collision?

Example 17, cont

Consider the block and bullet as a system. Initially the angular momentum is all in the bullet. Since there are no external torques on the system, the angular momentum is constant.

(a) Since $\sum \tau_{ext} = 0,\ L_f = L_i.\ L_i = mv\ell,\ L_f = (m+M)v_f\ell = mv\ell.$

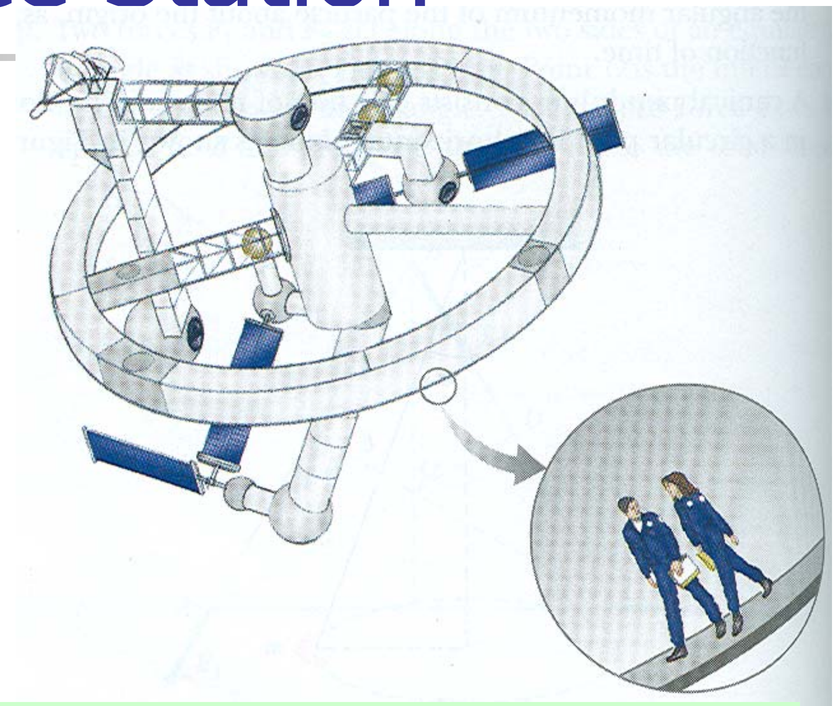
(b) $K_i = \frac{1}{2}mv^2,\ K_f = \frac{1}{2}(M+m)v_f^2$

$$v_f = \left(\frac{m}{m+M}\right)v,\ K_f = \frac{1}{2}mv^2\left(\frac{m}{M+m}\right)$$

$$\frac{|\Delta K|}{K_i} = \frac{K_i - K_f}{K_i} = \frac{1}{2}mv^2\left(1 - \frac{m}{M+m}\right)\left(\frac{1}{2}mv^2\right)^{-1} = \frac{M}{M+m}$$

Example 18: Space Station

A space station is constructed in the shape of a hollow ring of mass 5.00 x 10^4 kg. (Other parts make a negligible contribution to the total moment of inertia.) The crew walk on the inner surface of the outer cylindrical wall of the ring, with a radius of 100 m. At rest when constructed, the ring is set rotating about its axis so that people inside experience an effective free-fall acceleration equal to *g*. The rotation is achieved by firing two small rockets attached tangentially to opposite points on the outside of the ring.



(a) What angular momentum does the space station acquire?

(b) How long must the rockets fire if each exerts a thrust of 125 N?

Example 18, cont

$$\text{(a) Require } g = \frac{v^2}{r} = \omega^2 r. \quad \omega = \sqrt{\frac{g}{r}} = \sqrt{\frac{9.80\ \text{m/s}^2}{100\ \text{m}}} = 0.313\ \text{rad/s}$$

$$I = Mr^2 = 5.00\times10^4\ \text{kg}\ (100\ \text{m})^2 = 5.00\times10^8\ \text{kg m}^2$$

$$L = I\omega = \left(5\times10^8\ \text{kg m}^2\right)\left(0.313\ \text{rad/s}\right) = 1.57\times10^8\ \text{kg m}^2/\text{s}$$

$$\text{(b) } \tau = 2rF = 2\left(100\ \text{m}\right)\left(125\ \text{N}\right) = 2.50\times10^4\ \text{N m}$$

$$\tau = \frac{dL}{dt} = \frac{L_f - L_i}{\Delta t} = 2.50\times10^4\ \text{N m}$$

$$\Delta t = \frac{1.57\times10^8\ \text{kg m}^2/\text{s} - 0\ \text{kg m}^2/\text{s}}{2.50\times10^4\ \text{Nm}} = 6280\ \text{s}$$

The total torque on the ring, multiplied by the time interval found in part (b), is equal to the change in angular momentum, found in part (a). This equality represents *the angular impulse-angular momentum theorem*.